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TESSERAL MATRICES

By T. SMITH (*Teddington*)

[Received 15 June 1931]

1. When it is desired to exhibit the number π of rows and the number ρ of columns of a matrix u the notation ${}_\pi u^\rho$ will be employed. The orders of a product are then given by the external subscript and superscript: thus

$$w = uv = {}_\pi u^\rho {}_\rho v^\sigma = {}_\pi w^\sigma.$$

Suppose that p, r, s are definite non-singular square matrices of orders π, ρ, σ and that P, R, S are their inverses. From u, v, w we form new matrices U, V, W,

$$U = puR, \qquad V = rvS, \qquad W = pwS,$$

by prefixing the matrix corresponding to the order of the subscript and postfixing the inverse matrix corresponding to the order of the superscript. Then

$$W = puvS = puRrvS = UV; \qquad W' = V'U',$$

where accents denote conjugate matrices. It is evident that, if U' and V' are the inverses of u and v, then W' will be the inverse of w provided the sequence π, ρ, σ is monotonic.

2. Now suppose that the relation between u and U is symmetrical, so that

$$u = pUR = p^2 u R^2.$$

This equation will hold for a large class of matrices if p^2 and R^2 are scalar, or more precisely if $p = \epsilon P$, $r = \epsilon R$, where ϵ is a definite scalar. No loss of generality results from assuming $|\epsilon| = 1$. If $\epsilon = 1$, our condition leads to a group of which orthogonal matrices are the simplest examples. If $\epsilon = -1$, the simplest class is obtained by taking

$$p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where 0 and 1 denote zero and unit matrices of the same order. The orders of u are then necessarily even, and it is convenient to replace π, ρ, σ by 2π, 2ρ, 2σ. A typical matrix u of this class is related to its reciprocal U and its inverse U' as shown by the equations

$$u = \begin{pmatrix} b & d \\ a & c \end{pmatrix}, \qquad U = \begin{pmatrix} c & -a \\ -d & b \end{pmatrix}, \qquad U' = \begin{pmatrix} c' & -d' \\ -a' & b' \end{pmatrix}, \qquad (2.1)$$

where a, b, c, d are each of the form ${}_\pi q^\rho$. I shall refer to members of this class as tesseral matrices.

Tesseral matrices are necessarily non-singular, and the way in which they have been derived shows that the product of tesseral matrices is a tesseral matrix if the orders form a monotonic sequence.

3. In what follows we assume that at least one of the components a, b, c, d is non-singular. This is not a general property of tesseral matrices. For example,

$$u = U = \begin{pmatrix} 1 & . & . & . \\ . & . & . & -1 \\ . & . & 1 & . \\ . & 1 & . & . \end{pmatrix}$$

is a tesseral matrix having a, b, c, d all singular.

The four components a, b, c, d are not independent, and the connexion can be shown explicitly. Consider the case when a is non-singular and $\pi \leqslant \rho$. The appropriate condition $uU' = 1$ implies

$$bc' - da' = 1, \qquad cb' - ad' = 1, \qquad ac' = ca', \qquad bd' = db',$$

three independent relations, for the second is the conjugate of the first. The third condition can always be satisfied by

$$c = af, \qquad c' = fa',$$

where f is symmetrical of order ρ, and then the first condition if

$$d = bf - A,$$

where A is any* solution of $Aa' = 1$. Substitution in $bd' = db'$ gives $bA' = Ab'$, and the general solution is

$$b = AE, \qquad b' = EA',$$

where E is symmetrical.

The resulting expressions for u and U' are

$$u = \begin{pmatrix} AE & AEf - A \\ a & af \end{pmatrix}, \qquad U' = \begin{pmatrix} fa' & A' - fEA' \\ -a' & EA' \end{pmatrix},$$

from which it is clear that the conditions $uU' = 1$, $U'u = 1$ reduce to $aA' = 1$, $A'a = 1$ respectively.

We shall be concerned only with the slightly less general solution derived from the substitution

$$b = ea, \qquad b' = a'e, \qquad e = AEA'.$$

For convenience we shall call tesseral matrices of this type τ-matrices. The four forms for τ-matrices corresponding to non-singular values

* A is only determinable uniquely from a when $\pi = \rho$.

of a, b, c, d are

$$\tau = \begin{pmatrix} ea & eaf-A \\ a & af \end{pmatrix}, \qquad T' = \begin{pmatrix} fa' & A'-fa'e \\ -a' & a'e \end{pmatrix}, \tag{3.1}$$

$$\tau = \begin{pmatrix} b & bh \\ gb & gbh+B \end{pmatrix}, \qquad T' = \begin{pmatrix} hb'g+B' & -hb' \\ -b'g & b' \end{pmatrix}, \tag{3.2}$$

$$\tau = \begin{pmatrix} icj+C & ic \\ cj & c \end{pmatrix}, \qquad T' = \begin{pmatrix} c' & -c'i \\ -jc' & jc'i+C' \end{pmatrix}, \tag{3.3}$$

$$\tau = \begin{pmatrix} dl & d \\ kdl-D & kd \end{pmatrix}, \qquad T' = \begin{pmatrix} d'k & -d' \\ D'-ld'k & ld' \end{pmatrix}. \tag{3.4}$$

If $\tau = {}_{2\pi}\tau^{2\rho}$, then e, g, i, k are symmetrical of order π and f, h, j, l symmetrical of order ρ. The condition $\tau T' = 1$ depends upon $aA' = bB' = cC' = dD' = 1$ in these respective cases, and similarly $T'\tau = 1$ upon $A'a = B'b = C'c = D'd = 1$. When any of the components a, b, c, d is singular the corresponding τ form does not exist. No confusion will arise, if we also regard u in (2.1) as a τ-matrix, provided it can be put into one of the forms (3.1), (3.2), (3.3), (3.4).

4. We now proceed to extend the multiplication theorem by showing that the product of τ-matrices is a τ-matrix subject only to the obvious conditions concerning the rank of the product components. It will be sufficient to take the A form of τ as representative of the A and D forms, and the B form as representative of the B and C forms. Also we need only take the cases $\pi \leqslant \rho \leqslant \sigma$ and $\pi \leqslant \rho \geqslant \sigma$, $\pi \leqslant \sigma$ to represent the four cases that may arise. By direct multiplication we have

$$\begin{pmatrix} e_1a_1 & e_1a_1f_1-A_1 \\ a_1 & a_1f_1 \end{pmatrix}\begin{pmatrix} e_2a_2 & e_2a_2f_2-A_2 \\ a_2 & a_2f_2 \end{pmatrix} = \begin{pmatrix} ea & eaf-(e_1-e)a(f_2-f) \\ a & af \end{pmatrix}$$

identically, where

$$a = a_1(f_1+e_2)a_2, \tag{4.1}$$

$$(e_1-e)a = A_1a_2, \qquad a(f_2-f) = a_1A_2. \tag{4.2), (4.3}$$

The components of the product matrix are linear in the magnitudes represented by the compound symbols ea and af. We have to show that it is legitimate to represent them in this way as products, that e and f may then always be symmetrical, and that $(e_1-e)a(f_2-f)$ is reciprocal to a. The notation is clearly justified, as (4.2) and (4.3) can be solved for e and f.

If $\pi \leqslant \rho \leqslant \sigma$, the equations immediately yield

$$aA_2'a_1' = a_1(f_1+e_2)a_1', \text{ a symmetrical matrix, } = a_1A_2a',$$

$$(e_1-e)aA_2'a_1' = 1, \qquad a(f_2-f)a' = a_1A_2a'.$$

These conditions require e to be a symmetrical matrix of order π, and always permit f to be symmetrical of order σ. Moreover,

$$(e_1-e)a(f_2-f)a' = 1,$$

so that we are justified in writing

$$(e_1-e)a(f_2-f) = A,$$

which completes the proof for this case.

When $\pi \leqslant \rho \geqslant \sigma$, put $$\theta = a_2A'a_1. \tag{4.4}$$

From (4.1) we have

$$\theta(f_1+e_2)\theta = a_2A'aA'a_1 = \theta, \tag{4.41}$$

when either $aA' = 1$ or $A'a = 1$. This equation always admits of a symmetrical value of θ. When θ is known, A' is found from

$$A' = A_2'\theta A_1', \tag{4.42}$$

and

$$(e_1-e)a(f_2-f) = (e_1-e)aA'a(f_2-f) = A_1a_2A'a_1A_2 = A_1\theta A_2 = A.$$

If now $\pi \leqslant \sigma$,

$$aA' = 1 \quad \text{and} \quad e_1-e = A_1a_2A' = A_1\theta A_1', \tag{4.43}$$

so that e is necessarily symmetrical, and

$$a(f_2-f)a' = aA_2'\theta A_2a', \tag{4.44}$$

showing that a symmetrical form for f exists. This condition is clearly met if we take the solution

$$f_2-f = A_2'\theta A_2 = A'a_1A_2. \tag{4.45}$$

When $\pi = \rho = \sigma$, (4.43) and (4.45) represent the only solution which exists, and it is clear that they are always acceptable solutions of (4.2) and (4.3). We remove all ambiguity by invariably adopting these expressions for e and f.

Turning now to the form of τ involving B, we have

$$\begin{pmatrix} b_1 & b_1h_1 \\ g_1b_1 & g_1b_1h_1+B_1 \end{pmatrix}\begin{pmatrix} b_2 & b_2h_2 \\ g_2b_2 & g_2b_2h_2+B_2 \end{pmatrix}$$

$$= \begin{pmatrix} b & bh \\ gb & gbh+B_1B_2-(g-g_1)b(h-h_2) \end{pmatrix},$$

where
$$b = b_1(1+h_1g_2)b_2, \tag{4.5}$$
$$(g-g_1)b = B_1g_2b_2, \tag{4.6}$$
$$b(h-h_2) = b_1h_1B_2. \tag{4.7}$$

Taking first the case $\pi \leqslant \rho \leqslant \sigma$, (4.5) gives

$$\begin{aligned} bB_2'h_1b_1' &= b_1(h_1+h_1g_2h_1)b_1', \text{ a symmetrical matrix,} \\ &= b_1h_1B_2b' = b(h-h_2)b', \end{aligned}$$

so that a symmetrical value of h is possible. Also

$$(g-g_1)b(h-h_2)b' = B_1g_2h_1b_1' = B_1(g_2h_1+1)b_1'-1 = B_1B_2b'-1,$$

which justifies us in writing

$$B = B_1B_2-(g-g_1)b(h-h_2), \qquad Bb' = 1.$$

From this equation

$$\begin{aligned} Bb_2'g_2b_2 &= [B_1-(g-g_1)b_1h_1]g_2b_2 \\ &= (g-g_1)(b-b_1h_1g_2b_2) = (g-g_1)b_1b_2, \end{aligned}$$

or
$$Bb_2'g_2B_1' = g-g_1 = B_1g_2b_2B',$$

so that g is necessarily symmetrical.

Coming now to the case $\pi \leqslant \rho \geqslant \sigma$, put $\phi = b_2B'b_1$. Then

$$\phi = b_2B'bB'b_1 = \phi(1+h_1g_2)\phi, \tag{4.8}$$

showing that values of ϕ can be found which give symmetrical values of ϕh_1 and $g_2\phi$. It is not to be assumed that both are symmetrical for the same value of ϕ. Take $g_2\phi = \phi'g_2$, and multiply the equation for ϕ behind by $h_1\phi'$. Then

$$\begin{aligned} \phi^2h_1\phi' &= \phi h_1\phi'-\phi h_1 . g_2\phi . h_1\phi', \text{ a symmetrical matrix,} \\ &= \phi h_1\phi'^2, \end{aligned}$$

from which we can justify the equality

$$\phi h_1 = h_1\phi'.$$

Also, if $\pi \leqslant \sigma$,
$$g-g_1 = B_1g_2b_2B' = B_1g_2\phi B_1'$$
and
$$b(h-h_2)b' = bB_2'\phi h_1B_2b',$$

so that g is necessarily symmetrical, and a symmetrical value can be found for h. If we now assume that B satisfies the equation

$$B_1 = Bb_2'(1+g_2h_1),$$

we note that the conditions of order and rank are satisfied. Multiplication behind by b_1' gives

$$B_1b_1' = Bb',$$

by (4.5), multiplication in front by b_1' and behind by ϕ' gives

$$\phi' = \phi'(1+g_2h_1)\phi'$$

in accordance with (4.8), and $B_1g_2b_2B'$, i.e. $g-g_1$, is obviously symmetrical. We are therefore justified in adopting this equation as a definition of B. Now multiply the equation behind by B_2. Noting that $Bb_2' = B_1\phi'$, we have

$$B_1B_2 - B = B_1\phi'g_2h_1B_2 = B_1g_2\phi h_1B_2 = (g-g_1)b(h-h_2),$$

which completes the proof of the theorem.

As with the A form, we may always render the solution of (4.6) and (4.7) unambiguous by confining our attention to the particular solution

$$g-g_1 = B_1g_2b_2B', \tag{4.91}$$

$$h-h_2 = B'b_1h_1B_2. \tag{4.92}$$

5. When more than one form for a particular τ-matrix exists, relations must exist between their elements. Let us suppose $\pi \leqslant \rho$. Comparing the first columns of (3.1) and (3.2), we have

$$a = gb, \qquad b = ea, \qquad a = gea,$$

or

$$ge = 1 = eg,$$

on multiplying behind by A'.

Again, from the first columns of (3.1) and (3.3)

$$ea = ia + C,$$

or

$$e = i + CA'.$$

Finally, from the second columns of (3.1), (3.2), and (3.4),

$$A = eaf - d = e(gbh+B) - d = eB.$$

These relations are typical of a complete set involving e, g, i, k, viz.

$$A = eB, \qquad B = gA, \qquad C = iD, \qquad D = kC, \tag{5.1}$$

$$e-i = CA' = AC', \qquad k-g = BD' = DB', \tag{5.2}$$

$$eg = ge = 1, \qquad ik = ki = 1, \tag{5.3}$$

which also involve

$$ek - AD' = 1, \qquad gi + BC' = 1. \tag{5.4}$$

We next obtain equations connecting e, g, i, k with f, h, j, l. From (3.1) and (3.4)

$$Al = (ec-d)l = e(D+a) - dl = eD \tag{5.51}$$

and

$$fD' = f(b'k-a') = (A'+d')k - c' = A'k. \tag{5.52}$$

Similarly

$$gC = Bj, \qquad B'i = hC'. \tag{5.53}$$

Finally, considering the relations involving f, h, j, l, we obtain equa-

tions corresponding to (5.1)

$$Cf = (b - ia)f = A + d - ic = A$$

directly, or alternatively by (5.1), (5.3), and (5.51), (5.52), (5.53),

$$A = Cf, \qquad B = Dh, \qquad C = Aj, \qquad D = Bl. \tag{5.6}$$

When $\pi \geqslant \rho$ we may also establish relations corresponding to (5.2), (5.3), and (5.4), but when $\pi < \rho$ we can only infer lower-order relations such as $a(fj-1) = 0$, owing to the failure of a and af to specify f uniquely. When the ambiguities in the symmetrical factors of order ρ are avoided as in the last section, the limiting factors such as a can be removed and one form of the τ-matrix can be obtained from another. All the remaining equations can be derived from a suitable set of three. For example, noting that

$$A'B = (c'e - d')(c - gd) = (c' - d'g)(ec - d) = B'A$$

by (5.3), and similarly $C'D = D'C$, we see that the assumptions

$$f - h = A'B = B'A, \qquad l - j = C'D = D'C \tag{5.7}$$

are consistent with the symmetry of f, h, j, l. The first of these assumptions involves the equivalence of (3.1) and (3.2), and the second that of (3.3) and (3.4). To complete the connexion of the whole series, we assume $fj = 1$. From (5.7)

$$f(j + C'D) = (h + A'B)l$$

or

$$fj + A'D = hl + A'D$$

by (5.6), so that we have both

$$fj = jf = 1, \qquad hl = lh = 1. \tag{5.8}$$

By (5.6), (5.7), (5.8) we easily derive

$$fl - A'D = 1, \qquad hj + B'C = 1. \tag{5.9}$$

6. The elements of the standard τ-matrices (3.1), (3.2), (3.3), (3.4) which are absent from the u-matrix (2.1) may be arranged to form four symmetrical square matrices of order $\pi + \rho$. The definitions are

$$\left.\begin{aligned} \alpha &= \begin{pmatrix} e & -A \\ -A' & f \end{pmatrix}, \qquad \beta = \begin{pmatrix} g & B \\ B' & -h \end{pmatrix}, \\ \gamma &= \begin{pmatrix} i & C \\ C' & -j \end{pmatrix}, \qquad \delta = \begin{pmatrix} k & D \\ D' & l \end{pmatrix}. \end{aligned}\right\} \tag{6.1}$$

It will be noticed that the symmetry of α and δ is more complete than that of β and γ. This may be ascribed to the displacements

which have to be imparted to the elements of the second and third forms of τ to bring them into the positions they assume in β and γ.

By (5.4), (5.51), (5.52), (5.53), (5.9) we see that

$$\alpha\delta = 1 = \delta\alpha, \qquad \beta\gamma = 1 = \gamma\beta. \tag{6.2}$$

7. We now consider the bearing of the relations that have been found on contact transformations. In the first place we shall deal with the significance of the τ multiplication theorem, taking as an illustration the A form associated with the α-matrix, and afterwards extend the result to the other forms by changes of variables. Let x, y, z be matrices of variables having the same number of rows and π, ρ, σ columns respectively. As before, let the orders of τ_1 be 2π, 2ρ, of τ_2 be 2ρ, 2σ, and of τ be 2π, 2σ.

Now the conditions that identically

$$(x\ \ y)\alpha_1\begin{pmatrix}x'\\y'\end{pmatrix}+(y\ \ z)\alpha_2\begin{pmatrix}y'\\z'\end{pmatrix}-(x\ \ z)\alpha\begin{pmatrix}x'\\z'\end{pmatrix}$$
$$=(xA_1-p+zA_2')\theta(A_1'x'-p'+A_2z') \tag{7.1}$$

are
$$y=p\theta, \qquad \theta=\theta', \tag{7.2), (7.3}$$
$$\theta=\theta(f_1+e_2)\theta, \qquad A=A_1\theta A_2, \tag{7.31), (7.32}$$
$$e_1-e=A_1\theta A_1', \qquad f_2-f=A_2'\theta A_2. \tag{7.33), (7.34}$$

The last four equations correspond exactly to (4.41), (4.42), (4.43), (4.45), where we also assumed that (7.3) was satisfied. Thus, when $\pi\leqslant\rho\geqslant\sigma$, the left side can be factorized as shown on the right, provided the magnitudes of the elements in α are those which would be derived from the τ-equation $\tau=\tau_1\tau_2$.

If we substitute from (7.2) and (7.31) in (7.1), we may write the equation in the form

$$(x\ \ y)\alpha_1\begin{pmatrix}x'\\y'\end{pmatrix}+(y\ \ z)\alpha_2\begin{pmatrix}y'\\z'\end{pmatrix}-(x\ \ z)\alpha\begin{pmatrix}x'\\z'\end{pmatrix}$$
$$=(xAa_2'-y+zA'a_1)(f_1+e_2)(a_2A'x'-y'+a_1'Az'), \tag{7.4}$$

where the value of θ given in (4.4) has been inserted.

When $\pi\leqslant\rho\leqslant\sigma$ we consider the conditions that

$$(x\ \ y)\alpha_1\begin{pmatrix}x'\\y'\end{pmatrix}+(y\ \ z)\alpha_2\begin{pmatrix}y'\\z'\end{pmatrix}-(x\ \ z)\alpha\begin{pmatrix}x'\\z'\end{pmatrix}$$
$$=[x(e_1-e)a-qa+z](f_2-f)[a'(e_1-e)x'-a'q'+z']$$

should be an identity. As before, matrix multiplication of the τ's is indicated, together with $y = qa_1$. Since

$$(e_1-e)a_1 = (e_1-e)a(f_2-f)a_2' = Aa_2',$$

the identity may be expressed in the form

$$(x\ \ y)\alpha_1\begin{pmatrix}x'\\y'\end{pmatrix} + (y\ \ z)\alpha_2\begin{pmatrix}y'\\z'\end{pmatrix} - (x\ \ z)\alpha\begin{pmatrix}x'\\z'\end{pmatrix}$$
$$= (xAa_2'-y+zA'a_1)A_1'a(f_2-f)a'A_1(a_2A'x'-y'+a_1'Az'). \quad (7.5)$$

Now since the extreme factors in the products on the right sides of (7.4) and (7.5) are the conjugates of one another, the left sides and their differentials with respect to y vanish when

$$y = xAa_2'+zA'a_1. \quad (7.6)$$

Consider, then, the problem of evaluating the sum

$$(x\ \ y)\alpha_1\begin{pmatrix}x'\\y'\end{pmatrix} + (y\ \ z)\alpha_2\begin{pmatrix}y'\\z'\end{pmatrix},$$

where α_1 and α_2 are symmetrical matrices, and y is to receive the value which makes the sum stationary for any small variations in y. In general α_1 is a function of x, y, and α_2 a function of y, z. Obviously the result can be expressed in the form

$$(x\ \ z)\alpha\begin{pmatrix}x'\\z'\end{pmatrix},$$

where α is a symmetrical matrix in general depending on x and z. The equations (7.4) and (7.5) show that α is determinable from the matrix $\tau = \tau_1\tau_2$, provided the value of y given by (7.6) is inserted in the product. This equation for y is convenient for solution by successive approximation, and is very suitable for use with the matrices since the effective functions a_1 and a_2' are known. This form of solution has obvious advantages over that derived by a direct attack upon the problem.

8. Let x, y, ξ, η be matrices which at the moment are only restricted to having the same number of rows and π, ρ, π, ρ columns respectively. For convenience write also

$$\kappa = (x\ \ y), \qquad \lambda = (\xi\ \ -y), \qquad \mu = (x\ \ -\eta), \qquad \nu = (\xi\ \ -\eta) \quad (8.1)$$

and $$\mathbf{A} = \kappa\alpha\kappa', \qquad \mathbf{B} = \lambda\beta\lambda', \qquad \mathbf{C} = \mu\gamma\mu', \qquad \mathbf{D} = \nu\delta\nu'. \quad (8.2)$$

By (6.2) we have identically

$$(\kappa\alpha+\nu)(\kappa'+\delta\nu') = \mathbf{A}+\mathbf{D}+\kappa\nu'+\nu\kappa' \quad (8.3)$$

and $$(\lambda\beta+\mu)(\lambda'+\gamma\mu') = \mathbf{B}+\mathbf{C}+\lambda\mu'+\mu\lambda'. \quad (8.4)$$

Also, by various relations given in § 5,

$$(xe-yA'+\xi)(x'-By'+g\xi') = \mathbf{A}+\mathbf{B}+x\xi'+\xi x', \tag{8.5}$$

$$(xi+\xi-\eta C')(x'+k\xi'-D\eta') = \mathbf{D}+\mathbf{C}+x\xi'+\xi x', \tag{8.6}$$

$$(xA-yf+\eta)(C'x'-y'+j\eta') = \mathbf{A}-\mathbf{C}-y\eta'-\eta y', \tag{8.7}$$

$$(yh+\xi B-\eta)(y'+D'\xi'-l\eta') = \mathbf{D}-\mathbf{B}-y\eta'-\eta y'. \tag{8.8}$$

Now in all these six identities the two factors on the left are obtainable from one another by transposition and multiplication by a suitable factor. It follows that by making these factors vanish we obtain contact transformations involving the expressions on the right of these equations. It is easily verified that x, y, ξ, η are used consistently in these equations. Thus all the twelve left-side factors vanish if $\kappa\alpha+\nu = 0$ in virtue of the relations of § 5. We are therefore able to construct the remaining functions of the group **A**, **B**, **C**, **D** from a knowledge of any one of them, provided more than one exists. For example, given **A** as a function of x and y, we can determine **D** a function of ξ and η defined by

$$\mathbf{A}+\mathbf{D}+\kappa\nu'+\nu\kappa' = 0, \qquad \frac{d\mathbf{D}}{d\kappa} = 0$$

by substituting from $\kappa+\nu\delta = 0$ in δ. Transformations of this type exemplify the principle of duality.

The slightly imperfect symmetry of β and γ appears again in the transposition of the Greek and roman letters in passing from λ to μ. The negative signs in the second members of λ, μ, ν depend upon the definitions adopted for β, γ, δ. The signs in these expressions have been chosen for convenience and to enable the standard relations to include the simple equations (6.2).

To the foregoing relations we add an equation of a different type which is involved in $\kappa+\nu\delta = 0$, viz.

$$(y\ -\eta) = (x\ -\xi)\tau \tag{8.9}$$

and its inverse

$$(y\ -\eta)T' = (x\ -\xi). \tag{8.91}$$

This equation is suitable for use when τ is given initially as a function of x and ξ. It shows that the problems which give rise to these methods can be solved in a different way. It should be remembered that the relations which subsist between the elements of tesseral matrices only involve their magnitudes, and a particular set of magnitudes which is here always represented by a single character may in different equations be different functions of different variables.

9. Expressions which involve both the small and capital roman letters, such as (7.6), are formally indeterminate when the components are not square. The difficulty is almost always only apparent. In this example we may adopt xA and zA' as the effective variables. It will almost invariably happen, however, that the final product of the τ's is a square matrix, and there is no ambiguity in the expressions ultimately obtained. The initial conditions also usually give rise to square matrices. Components with unequal orders are frequently the result of simplification in the early stages of a problem.

Some elementary results of importance may be noted.

(i) If $b = c = 1$ and either $a = 0$ or $d = 0$, the matrix is a τ-matrix.

(ii) If $\begin{pmatrix} b_1 & d_1 \\ a_1 & c_1 \end{pmatrix}$ and $\begin{pmatrix} b_2 & d_2 \\ a_2 & c_2 \end{pmatrix}$ are tesseral matrices multiplying on the same side with their inverses to yield scalars, so is

$$\begin{pmatrix} b_1 & . & d_1 & . \\ . & b_2 & . & d_2 \\ a_1 & . & c_1 & . \\ . & a_2 & . & c_2 \end{pmatrix},$$

and if the first two are τ-matrices, so is the third. The converse of this proposition is also true.

(iii) If $\begin{pmatrix} b & d \\ a & c \end{pmatrix}$ is a tesseral matrix, so is the matrix obtained by deleting any corresponding long rows* from a, b, c, d. Since the distinguishing property of a τ-matrix is that the long rows of at least one of the four components a, b, c, d are independent, we see that the contracted matrix is a τ-matrix, if the original matrix is a τ-matrix, but the converse of this proposition is not true.

(iv) The deletion of any long rows from a, b, c, d will be accompanied by the deletion of the corresponding rows from A, B, C, D.

(v) Only two components of a tesseral matrix may have corresponding long zero rows, and these must be a diagonal pair.

(vi) Corresponding short zero rows may be added to or deleted from the four constituents a, b, c, d without affecting the character of a tesseral or τ-matrix.

(vii) In general, short rows may not be removed from tesseral matrices.

* i.e. either rows or columns, the choice depending on which has the greater number of elements. The other set of rows are the short rows.

A PROPERTY OF INTEGRAL FUNCTIONS OF FINITE ORDER

By J. M. WHITTAKER (*Cambridge*)

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1. A WELL-KNOWN theorem due to Littlewood, Wiman, and Valiron* states that an integral function of order less than a half is 'flat', i.e.

$$\log m(r) > \text{a positive constant} \times \log M(r),$$

on a sequence of circles of indefinitely increasing radius. In a recent paper† is was shown that these circles can be broadened out into annuli of flatness; in fact that

$$\log m_\sigma(r) > \text{a positive constant} \times \log M_\sigma(r)$$

for a sequence of annuli $r \leqslant |z| \leqslant r+r^\sigma$, $(\sigma < 1-\rho)$, $m_\sigma(r)$, $M_\sigma(r)$ denoting the bounds of $|f(z)|$ in the annulus.

Functions of higher order need not possess annuli of flatness, but it was shown in I that in the case of a function of genus 0 or 1‡ there is a sequence of circles of fixed radius within which the function is flat. Thus,

THEOREM A. *Let $f(z)$ be an integral function of genus* 0 *or* 1 *and let d be a given positive number. Then there is a positive constant h and a sequence $\zeta_1, \zeta_2, \ldots$ such that*

$$\lambda_s = |\zeta_s| \to \infty \tag{1}$$

and

$$\log|f(z)| > h \log M(\lambda_s) \tag{2}$$

in the circle $|z-\zeta_s| \leqslant d$.

* Littlewood, *Proc. London Math. Soc.*, 6 (1908), 189–204; Wiman, *Math. Ann.*, 76 (1915), 197–211; Valiron, *Ann. de Toulouse*, 5 (1913), 117–257. Proofs have also been given by Pólya, *J. London Math. Soc.*, 1 (1926), 78–86; and Besicovitch, *Math. Ann.*, 97 (1927), 677–95.

† *Proc. Edinburgh Math. Soc.*, 2 (1930), 111–28. This will be referred to as I. Miss Cartwright has pointed out to me that Theorem 2 of I, on functions with zeros at the lattice points, can be deduced from a result of F. and R. Nevanlinna, 'Über die Eigenschaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie', *Acta Soc. Scient. Fennicae*, 50 (1922), p. 42.

‡ i.e. such that

$$\int_1^\infty \frac{\log M(x)}{x^3}\,dx$$

converges. See Valiron, *Integral Functions*, Toulouse (1923), p. 185.

In the present paper a result of this type applicable to functions of any finite order is given.

THEOREM C. *Let $f(z)$ be an integral function of order ρ, and let σ be a given number $< 1-\frac{1}{2}\rho$. Then there is a positive constant h and a sequence $\zeta_1, \zeta_2, \ldots$ such that* (1) *is true and* (2) *is true in the circle*

$$|z-\zeta_s| \leqslant \lambda_s^{\sigma}. \tag{3}$$

This result is the best possible in the sense that σ cannot in general be replaced by $1-\frac{1}{2}\rho$ in (3).

2. Theorem C is suggested by writing $z = w^{\frac{1}{2}\rho}$ in Theorem A. Of course this procedure does not constitute a proof, but the demonstration which follows can be looked on as a rigorous formulation of the same principle. The details are rather long and have been omitted for the most part.

LEMMA 1. *Let $f(z)$ be of genus 0 or 1, and let λ, ϵ be given positive numbers. Divide the z-plane into squares of side λ by drawing lines parallel to the axes. Then for almost every square*

$$|\log f(z_1) - \log f(z_2)| < \epsilon r (\log r)^{\frac{1}{2}}, \tag{4}$$

where z_1, z_2 are any points of the square and r is the distance of the centre of the square from the origin.

This is essentially Lemma 6 of I. The conclusion of the lemma means that

$$\frac{N_s}{N} \to 0, \text{ as } N \to \infty,$$

where N_s is the number of squares, among the N nearest the origin, for which (4) is false.

LEMMA 2. *Let $f(z)$ be an integral function of the form*

$$Cz^l \times (\text{a canonical product of genus } p) \times \exp(b_1 z + b_2 z^2 + \ldots + b_q z^q), \tag{5}$$

having no zeros in the sector $\alpha < \theta < \beta$, $(z = re^{i\theta})$. Then, if

$$\alpha + 2\gamma \leqslant \theta \leqslant \beta - 2\gamma \text{ and } |h| < r \sin 2\gamma,$$

$$|\log f(z+h) - \log f(z)| < A_{\gamma,p}|h|r^p \sum_{n=1}^{\infty} \frac{1}{r_n^p (r_n + r)} + |h| \sum_{s=1}^{q} s|b_s| r^{s-1} +$$

$$+ \frac{|h|^{p+2}}{p+2} \operatorname{cosec}^{p+2}\gamma \sum_{n=1}^{\infty} \frac{1}{(r_n+r)^{p+2}} + \frac{|h|^{p+3}}{p+3} \operatorname{cosec}^{p+3}\gamma \sum_{n=1}^{\infty} \frac{1}{(r_n+r)^{p+3}} + \ldots \tag{6}$$

where $A_{\gamma,p}$ depends only on γ, p and $r_1, r_2, \ldots$ denote the moduli of the zeros $a_1, a_2, \ldots$ of $f(z)$.

We make use of the inequalities

$$|a_n+z| \geqslant (r_n+r)\sin\gamma, \qquad |a_n+z| \geqslant r\sin 2\gamma > h, \tag{7}$$

which are readily proved with the aid of a diagram. Write

$$E(u,p) = (1-u)\exp\left(u+\frac{u^2}{2}+\ldots+\frac{u^p}{p}\right),$$

so that

$$\log E\left(-\frac{z+h}{a_n}, p\right) - \log E\left(-\frac{z}{a_n}, p\right)$$

$$= \log\left(1+\frac{h}{a_n+z}\right) - \frac{z+h}{a_n}+\ldots+(-)^p\frac{(z+h)^p}{pa_n^p}+\frac{z}{a_n}-\ldots-(-)^p\frac{z^p}{pa_n^p},$$

and, on expanding the logarithm and making use of (7), we arrive at (6).

LEMMA 3. *Let $S_{m,n}$ be a mesh formed by the intersections of the hyperbolae*

$$x^2-y^2 = m, \qquad 2xy = n, \qquad x^2-y^2 = m+1, \qquad 2xy = n+1, \tag{8}$$

where m, n are positive integers, lying entirely within the domain $D_{R,\alpha}$ defined by the inequalities

$$\alpha \leqslant \theta \leqslant \tfrac{1}{4}\pi-\alpha, \ \tfrac{1}{2}R \leqslant r \leqslant R, \ (x+iy = re^{i\theta}, \ 0<\alpha<\tfrac{1}{8}\pi, \ R \geqslant 1). \tag{9}$$

Then there are constants A, B, C depending only on α such that

(i) $AR^{-2} \leqslant$ *area of* $S_{m,n} \leqslant BR^{-2}$,

(ii) $S_{m,n}$ *contains a circle of radius* CR^{-1}.

With the aid of a diagram this is readily seen to be true. The hyperbolae are the transforms of the straight lines of Lemma 1 (with $\lambda = 1$) when the transformation $z^2 = w$ is made. Similar results hold for the transformation $z^k = w$ where k is any integer. The hyperbolae are replaced by the curves

$$r^k\cos k\theta = m, \qquad r^k\sin k\theta = n, \qquad r^k\cos k\theta = m+1, \qquad r^k\sin k\theta = n+1,$$

and the domain $D_{R,\alpha}$ by the domain

$$\alpha \leqslant \theta \leqslant \frac{\pi}{2k}-\alpha, \qquad \tfrac{1}{2}R \leqslant r \leqslant R, \qquad (0<\alpha<\frac{\pi}{4k}, \ R \geqslant 1).$$

The meshes contain circles of radius CR^{1-k}.

3. We next consider the 'surface-density' of $\log|f(z)|$.

For a domain D of area $A(D)$ this may be defined to be

$$\sigma(f,D) = \frac{1}{A(D)}\iint_D \log|f(z)|\,dxdy.$$

It is shown in I (4.4) that for a circle of radius R this becomes

$$\sigma(R) = \log|C| + \tfrac{1}{2}l\log\frac{R}{e} + \int_{r_1}^{R} \frac{n(u)-l}{u}\left(1-\frac{u^2}{R^2}\right)du, \tag{10}$$

where $n(u)$ denotes the number of zeros of modulus not greater than u, and $f(z)$ is given by (5). In the case of an annulus $\frac{1}{2}R \leqslant r \leqslant R$ the surface-density may be written

$$\tau(R) = \frac{4}{3\pi R^2}\{\pi R^2\sigma(R) - \tfrac{1}{4}\pi R^2\sigma(\tfrac{1}{2}R)\}, \tag{11}$$

and, since, by (10), $\sigma(R)$ is an increasing function of R,

$$\sigma(R) \leqslant \tau(R). \tag{12}$$

Thus $\sigma(R)$ can be replaced by $\tau(R)$ in Theorem 5 of I, which is then as follows:

LEMMA 4. *If $f(z)$ is an integral function of finite non-integral order,*

$$\overline{\lim_{R\to\infty}}\frac{\tau(R)}{\log M(R)} > 0.$$

If $f(z)$ is of finite integral order, this property holds for the function $\{f(z)+d\}$ for every value of d except, possibly, one value.

4. We have now collected enough material to prove a preliminary theorem.

Let $f(z)$ be an integral function of genus p, order ρ. It is convenient to distinguish six cases as follows, i denoting a positive integer or zero:

(i) $p = 2i$, $\rho = 2i$. (ii) $2i < \rho < 2i+1$. (iii) $p = 2i$, $\rho = 2i+1$.
(iv) $p = 2i+1$, $\rho = 2i+1$. (v) $2i+1 < \rho < 2i+2$.
(vi) $p = 2i+1$, $\rho = 2i+2$.

LEMMA 5. *If $f(z)$ falls under one of cases* (i), (ii), (v), (vi), *then* (2) *is true in a sequence of circles*

$$|z-\zeta_s| \leqslant d\lambda_s^{-i}$$

It will appear later that the lemma is true in cases (iii), (iv), but the proof is different.

Consider first case (v) with $i = 1$, so that $3 < \rho < 4$. Let

$$\tau_\alpha(R) = \frac{1}{12\pi R^2}\int_{\frac{1}{2}R}^{R}\int_{\alpha}^{\alpha+\frac{1}{2}\pi} \log|f(ue^{i\theta})|\,u\,dud\theta$$

denote the surface-density in the domain

$$\tfrac{1}{2}R \leqslant r \leqslant R, \qquad \alpha \leqslant \theta \leqslant \alpha + \tfrac{1}{8}\pi.$$

Then for at least one of the values $\frac{1}{16}\pi$, $\frac{3}{16}\pi$, $\frac{5}{16}\pi$,... of α

$$\overline{\lim_{R\to\infty}} \frac{\tau_\alpha(R)}{\log M(R)} > 0. \tag{13}$$

There is no loss of generality in taking this value to be $\frac{1}{16}\pi$. We can write $f(z)$ in the form

$$f(z) = f_1(z)f_2(z),$$

where $f_1(z)$ is a canonical product with zeros b_1, b_2,... all of positive real part, and $f_2(z)$ is an integral function with zeros c_1, c_2,... all of zero or negative real part. Write

$$f_1(z)f_1(-z) = g(z^2),$$

so that $g(w)$ is an integral function of genus 1 of the complex variable $w = u+iv$, with zeros at the points $w = b_1^2$, b_2^2,... .

Now take positive numbers λ, ϵ and divide the w-plane into squares of side λ. Shade the squares for which the inequality

$$|\log g(w_1) - \log g(w_2)| < \epsilon t(\log t)^{\frac{1}{2}}$$

is true; w_1, w_2 are any points in a square whose centre is distant t from the origin.

Next divide the sector $0 \leqslant \theta \leqslant \frac{1}{4}\pi$ of the z-plane into meshes formed by the intersections of the hyperbolae

$$x^2-y^2 = m\lambda, \qquad 2xy = n\lambda, \qquad x^2-y^2 = (m+1)\lambda, \qquad 2xy = (n+1)\lambda,$$

and shade those meshes which correspond (by the transformation $z^2 = w$) to shaded squares in the w-plane. In a shaded mesh

$$|\log g(z_1^2) - \log g(z_2^2)| < 2\epsilon r^2(\log r)^{\frac{1}{2}} \qquad (r^2 = t),$$

and, applying Lemma 2 to the functions $f_2(z)$, $f_1(-z)$ and remembering that $|z_1-z_2| = O(r^{-1})$, it is found that

$$|\log f_2(z_1) - \log f_2(z_2)| = O(r^2),$$

a similar inequality holding for $f_1(-z)$. Now

$$f(z) = g(z^2)f_2(z)/f_1(-z),$$

so that in a shaded mesh

$$|\log f(z_1) - \log f(z_2)| < 3\epsilon r^2(\log r)^{\frac{1}{2}} \qquad (r \geqslant r_0). \tag{14}$$

Now it has been shown that there is a constant K_1 such that

$$\tau_\alpha(R) > K_1 \log M(R) \qquad (\alpha = \tfrac{1}{16}\pi), \tag{15}$$

for a sequence R_1, R_2,... of values of R.

By Lemma 3 the meshes in the domain

$$\tfrac{1}{2}R_s \leqslant r \leqslant R_s, \qquad \tfrac{1}{16}\pi \leqslant \theta \leqslant \tfrac{3}{16}\pi$$

are, roughly speaking, of the same size, and by Lemma 1 almost every one is shaded. The argument can now be completed as in the proof of Theorem A in I. Since, by (15), the average value of $\log|f(z)|$ is of the same order as $\log M(R_s)$, it follows that there must be at least one point z_1 in a shaded mesh such that

$$\log f(z_1) > K_2 \log M(R_s) > K_3 R_s^{\rho(R_s)} \qquad \big(\rho(R_s) > 3\big),$$

and so by (14) the inequality

$$\log f(z_2) > \frac{K_2}{2} \log M(R_s)$$

must be satisfied for all points z_2 in the mesh. Finally, given d, it follows from (ii) of Lemma 3 that λ can be so chosen that each mesh contains a circle of radius dR_s^{-1}.

This completes the proof of case (v). Cases (i), (ii), (vi) for $i = 1$ can be proved in much the same way.

The argument can be extended to prove the lemma for $i = 2, 3, 4, \ldots$. The appropriate transformation is $z^{i+1} = w$, and Lemma 3 is replaced by the extension of it involving the curves

$$r^{i+1}\cos(i+1)\theta = m, \qquad r^{i+1}\sin(i+1)\theta = n.$$

The auxiliary function $g(z)$ is defined by

$$g(z^{i+1}) = f_1(z) f_1(\omega z) \ldots f_1(\omega^i z),$$

where $1, \omega, \ldots, \omega^i$ are $(i+1)$th roots of unity and $f_1(z)$ is a canonical product having the same zeros as $f(z)$ in the sector

$$-\frac{\pi}{i+1} < \theta < \frac{\pi}{i+1}.$$

5. We can now enunciate a more precise result* of the same character.

THEOREM B. *Let $f(z)$ be an integral function of genus p. Then* (2) *is true in a sequence of circles*

$$|z - \zeta_s| \leqslant d\lambda_s^{(1-p)/2}.$$

As before, d is any given positive number.

As regards cases (v), (vi) this has been established in Lemma 5.

* This result has been made considerably sharper in accordance with a suggestion of Miss Cartwright.

Next take case (i). In this case the function $f(z^2)$ is of genus $4i$, order $4i$, and so is flat in a sequence of circles

$$|z-\zeta_s| \leqslant d\lambda_s^{-2i}.$$

or
$$|z^2-\zeta_s^2| \leqslant d\lambda_s^{-2i}|z+\zeta_s| = 2d\lambda_s^{-2i+1}+O(\lambda_s^{-4i}).$$

Writing $z^2 = w$, $\zeta_s^2 = \gamma_s$ this becomes

$$|w-\gamma_s| \leqslant 2d|\gamma_s|^{-i+\frac{1}{2}}+O(|\gamma_s|^{-2i}).$$

This region contains the circle

$$|w-\gamma_s| \leqslant d|\gamma_s|^{-i+\frac{1}{2}};$$

and $f(w)$ is therefore flat in this circle.

The same argument disposes of cases (iii), (iv), (ii).

6. Let $f(z)$ be a function of finite order ρ, and let $\sigma < 1-\frac{1}{2}\rho$. Write $\sigma = 1-\frac{1}{2}\rho'$ so that $\rho < \rho'$, and take an integer j such that

$$j > \frac{2}{\rho'-\rho}.$$

Then there is an integer i such that

$$\frac{2i}{j} \leqslant \rho < \frac{2i+2}{j}.$$

Also,
$$\rho' = \rho+(\rho'-\rho) > \frac{2i}{j}+\frac{2}{j} = \frac{2i+2}{j}.$$

Thus $f(z^j)$ is of order ρj, and so, by Theorem B, is flat in a sequence of circles

$$|z-\zeta_s| \leqslant d\lambda_s^{-i}.$$

Making the transformation $z^j = w$, we deduce that (2) is satisfied in a sequence of circles

$$|z-\zeta_s| \leqslant d\lambda_s^{(j-1-i)/j}$$

and the index of λ_s is greater than σ. This completes the proof of Theorem C.

7. It is perhaps worth while to add that if γ is any limit-point of the arguments of the points ζ_s, the radius $\theta = \gamma$ has the property that any sector containing it contains an infinite set of the points ζ_s. By Bolzano's theorem there must be at least one such radius. This is true both for Theorem B and for Theorem C.

FURTHER STUDIES OF EMDEN'S AND SIMILAR DIFFERENTIAL EQUATIONS

By R. H. FOWLER (*Cambridge*).

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1. Introduction. In a recent paper,* with perhaps some astrophysical applications, I have discussed in some detail the nature of the solutions of Emden's equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0 \qquad (n>1) \tag{1}$$

as $\xi \to 0$, or of the equation

$$\theta''+x^\sigma\theta^n=0 \qquad (n>1) \tag{2}$$

as $x \to \infty$, which is equivalent to (1) when $\sigma=-4$. This work arose out of my recent recognition, thanks to Professor Milne, of the astrophysical interest of continuing some much earlier incomplete work of my own on the nature of the solutions of the similar equation†

$$\frac{d}{dx}(x^\rho\theta')+bx^\sigma\theta^n=0 \qquad (n>1) \tag{3}$$

as $x \to \infty$. Only integral values of n are contemplated in I, but most of the arguments transfer immediately to rational values. The number of different cases presented by these equations in which the possible behaviour of the solutions differs and for which rather different arguments must be used is quite considerable, and the work quoted has not succeeded in giving an exhaustive analysis of the possibilities of every case. I attempt, I believe with success, in this third paper to fill in the main gaps, concluding it by repeating the complete results of all three papers in tabular form similar to that adopted in I. I do not seriously care whether these further investigations can be defended as being of physical or astrophysical interest; a resolution of the grouping and behaviour of the solutions of these simple non-linear equations may perhaps be found of sufficient intrinsic interest. But it is in fact likely that a detailed study such as this will prove of value in more than one physical problem as time goes on and the more difficult problems, which lead to non-linear

* R. H. Fowler, *M.N.R.A.S.*, vol. 91 (1930), p. 63. Quoted as II.

† R. H. Fowler, *Quart. J. of Math.*, vol. 45 (1914), p. 289, with references to earlier work. Quoted as I.

equations of this type, come more into the foreground. Such a problem, for example, as the distribution of ions in and around an electric arc, allowing for recombination, seems to lead to an equation of this non-linear type.*

2. Reduction of equation (3) to simpler forms. The earlier work in I refers to equation (3), but it is unnecessary to discuss an equation quite so general. (i) Suppose $\rho > 1$ and let

$$dx/x^\rho = dq/q^2, \qquad q = (\rho-1)x^{\rho-1}, \qquad \theta = y/q.$$

Then equation (3) becomes

$$y''+bq^{\sigma'}y^n = 0 \qquad (\sigma' = (\sigma+\rho)/(\rho-1)-n-3), \tag{4}$$

and $q \to \infty$ when $x \to \infty$. (ii) Suppose $\rho < 1$, and let

$$\frac{dx}{x^\rho} = dq, \qquad q = (1-\rho)x^{1-\rho}.$$

Then equation (3) becomes

$$\theta''+bq^{\sigma''}\theta^n = 0 \qquad \big(\sigma'' = (\sigma+\rho)/(1-\rho)\big), \tag{5}$$

and $q \to \infty$ when $x \to \infty$. The constants b in (4) and (5) have the same sign as the b in (3). (iii) Finally, if $\rho = 1$, let $q = \log x$. Then equation (3) becomes

$$\theta''+be^{\lambda q}\theta^n = 0 \qquad (\lambda = \sigma+1). \tag{6}$$

In order to discuss all cases covered by equation (3) it is therefore sufficient to discuss equation (2) together with the equation

$$\theta''+e^{\lambda x}\theta^n = 0 \qquad (n > 1). \tag{7}$$

We shall here be concerned only with equation (2). The omission of the factor b in (2) and (7) can be arranged for by a simple change of scale of θ, and is therefore trivial, but the sign of b matters. It is therefore necessary to include for completeness a discussion of

$$\theta''-x^\sigma\theta^n = 0 \qquad (n > 1), \tag{8}$$

$$\theta''-e^{\lambda x}\theta^n = 0 \qquad (n > 1). \tag{9}$$

* For studies of these equations reference must also be made to *Emden's Gaskugeln*, especially Chap. XII where Emden discusses the general solution of his equation, not merely the special solution associated with his name. His results, while not, I think, complete, contain a fair proportion of the theorems given here, obtained in a rather different form. Where his results and the results of this paper overlap, they agree as they should.

Mention should also be made of the work of Hopf, *M.N.R.A.S.* vol. 91 (1931), p 653, in which he proves the theorems of II and many of those of I by reducing the problem to the study of an equivalent equation of the first order.

We shall not discuss (9). Almost all these equations have already been adequately discussed in I when $n = 1$, and this case will not be touched on here.

In all cases, what is required is a full description of the possible behaviour of real continuous solutions of these equations, possessing continuous first differential coefficients as $x \to \infty$. Such solutions will be called *proper solutions*. It will be assumed that $n = p/q$, where p, q are integers relatively prime. The parity of p and q is important in so far as it controls the behaviour of θ^n as θ crosses or tries to cross $\theta = 0$. If p, q are both odd we shall be able to take $(-\theta)^n = -\theta^n$ and to say that n is *odd*. If p is even and q is odd, $(-\theta)^n = \theta^n$, and we say that n is *even*. If p is odd and q is even, $(-\theta)^n$ cannot be real and proper solutions cannot ever cross $\theta = 0$, still less oscillate about $\theta = 0$. This case is of rather less interest (mathematically). The chief interest lies in the equations (2) and (7) rather than (8) and (9) and in n odd rather than in n even, for it is only for (2) and (7) and n odd that oscillating solutions can occur.

3. Outline of known results. Detailed statements of the behaviour of the solutions in all cases will be given at the end of this paper. It will perhaps be best, however, to start with a simplified description of known results for equation (2). It has been shown in I and II that the nature of the solutions of equation (2) is largely controlled by the signs of the two expressions $\sigma+2$ and $\sigma+n+1$. If $\sigma+2 = 0$ or $\sigma+n+1 = 0$, equation (2) can be reduced to

$$\frac{d^2\theta}{dt^2} - \frac{d\theta}{dt} + \theta^n = 0 \qquad (t = \log x), \tag{10}$$

or

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y^n = 0 \qquad \left(t = \log x,\ y = \frac{1}{x}\theta\right), \tag{11}$$

respectively. The first equation has been fully discussed in I, see Table VI there; all its solutions must oscillate as $t \to \infty$, with oscillations of a specified form. The second has been fully discussed in II; all its solutions are ultimately monotonic as $t \to \infty$, again with specified forms, and specified relative arrangements. There remain the three cases of inequality:

(i) If $\sigma+2 > 0$, then $\sigma+n+1 > 0$ *a fortiori*. This case is fully discussed in I. We may summarize by saying that the behaviour is similar to that of $\sigma+2 = 0$.

(ii) If $\sigma+n+1 < 0$, then $\sigma+2 < 0$ *a fortiori*. This is a case of

astrophysical interest, fully discussed in II. The behaviour is similar to that of $\sigma+n+1=0$.

(iii) $\sigma+2<0$, $\sigma+n+1>0$. This is the most intricate and mathematically most interesting case. It is our main subject of study here. It subdivides into three according as $2\sigma+n+3<, =, >0$. It may sometimes possess simultaneously solutions which oscillate and others which are monotonic and tend to zero as $x\to\infty$.

4. The equation

$$\theta''+x^\sigma\theta^n=0 \quad (\sigma+n+1>0,\ \sigma+2<0,\ n>1,\ n \text{ odd}); \qquad \text{(A)}$$

various transformations and notation. Certain equivalent forms of equation (A) are here of greater assistance. On substituting

$$\theta=\lambda x^\varpi \qquad (\lambda,\ \varpi \text{ constant}),$$

we find that there are two special solutions of the form

$$\theta=\pm[\varpi(1-\varpi)]^{\frac{1}{n-1}}x^\varpi \qquad \left(\varpi=-\frac{\sigma+2}{n-1}\right). \qquad (12)$$

These are proper solutions since $0<\varpi<1$. We therefore write

$$\theta=[\varpi(1-\varpi)]^{\frac{1}{n-1}}x^\varpi v, \qquad (13)$$

and form the differential equation for v, which is

$$x^2v''+2\varpi xv'-\varpi(1-\varpi)(v-v^n)=0. \qquad (14)$$

We can further eliminate x from (14) by writing $t=\log x$, when (14) becomes

$$v''+(2\varpi-1)v'-cv(1-v^{n-1})=0 \qquad (c=\varpi-\varpi^2). \qquad (15)$$

The special solutions are now $v=\pm1$, and, of course, $v=0$. No other solution can touch these, for it would then be identical with the solution touched.

In (15) dashes denote, of course, differentiation with respect to t, and we shall always use them with this meaning relative to whichever independent variable is in use at the moment. We shall retain throughout the standardized relationships

$$x=e^t=1/\xi \qquad (t, x\to\infty,\ \xi\to0), \qquad (16)$$

$$\theta=xy=e^t\dot{y}=[\varpi(1-\varpi)]^{\frac{1}{n-1}}x^\varpi v=[\varpi(1-\varpi)]^{\frac{1}{n-1}}e^{\varpi t}v. \qquad (17)$$

The sign of $2\varpi-1$ changes when $\varpi=\frac{1}{2}$, i.e. when

$$2\sigma+n+3=0. \qquad (18)$$

The astrophysical example is $\sigma=-4$, $n=5$, first investigated by

Schuster. The discussion of this case is elementary and the results more or less known, but a short survey from the present viewpoint will serve to introduce the general cases

$$2\varpi-1>0, \qquad 2\sigma+n+3<0,$$
$$2\varpi-1<0, \qquad 2\sigma+n+3>0.$$

5. Equation (A). The special case $2\sigma+n+3=0$. We have to discuss the (v,t) equation

$$v''=\tfrac{1}{4}(v-v^n) \qquad (t\to\infty). \tag{19}$$

This integrates to give

$$\tfrac{1}{2}v'^2=\tfrac{1}{8}v^2-\frac{1}{4(n+1)}v^{n+1}+D. \tag{20}$$

Since $n>1$, $v\to\pm\infty$ implies $v'^2\to-\infty$, and so is impossible. We can write

$$\frac{dv}{\pm\left\{2D+\tfrac{1}{4}v^2-\dfrac{1}{2(n+1)}v^{n+1}\right\}^{\frac{1}{2}}}=dt \tag{21}$$

and v can at most oscillate to and fro between the greatest and least roots of

$$2D+\tfrac{1}{4}v^2-\frac{1}{2(n+1)}v^{n+1}=0. \tag{22}$$

When $D=0$,

$$\frac{dv}{v\left(1-\dfrac{2}{n+1}v^{n-1}\right)^{\frac{1}{2}}}=-\tfrac{1}{2}dt, \tag{23}$$

where the ambiguous sign has been chosen so that $t\to\infty$. With the help of the substitution

$$\frac{2}{n+1}v^{n-1}=\sin^2\lambda,$$

this is easily integrated in the form

$$\frac{2}{n+1}v^{n-1}=\frac{4C^2e^{-\frac{1}{2}(n-1)t}}{\{1+C^2e^{-\frac{1}{2}(n-1)t}\}^2}, \tag{24}$$

C being a constant of integration. The corresponding curves in other coordinates are

$$v=\pm\left[\frac{2(n+1)C^2x^{\frac{1}{2}(n-1)}}{\{x^{\frac{1}{2}(n-1)}+C^2\}^2}\right]^{\frac{1}{n-1}}, \tag{25}$$

$$\theta=\pm\left[\frac{\frac{1}{2}(n+1)C^2}{\{1+C^2/x^{\frac{1}{2}(n-1)}\}^2}\right]^{\frac{1}{n-1}}, \tag{26}$$

$$\theta=\pm\left[\frac{\frac{1}{2}(n+1)C^2}{\{1+C^2\xi^{\frac{1}{2}(n-1)}\}^2}\right]^{\frac{1}{n-1}}. \tag{27}$$

These solutions are finite at the origin and are therefore of Emden's type. They were, of course, first given by Schuster for $n = 5$. They are monotonic everywhere, decreasing as ξ increases, and tend to zero like A/ξ as $\xi \to \infty$, $x \to 0$.

The (v, t) Schuster-Emden curves form a one-parameter family with an asymptote easily shown to be

$$\pm v = \left(\frac{n+1}{2}\right)^{\frac{1}{n-1}} > 1.$$

The exceptional solutions $v = \pm 1$ therefore lie in the region covered by the Schuster-Emden solutions, which in (v, t) coordinates are all the same curve with different origins for t. Since, as is easily proved, $v'(t) > 0$ for $t < t_{\max}$ and $v'(t) < 0$ for $t > t_{\max}$, a pair of Schuster-Emden curves intersect once and once only. They therefore form a grid with the necessary properties for use in analysing the other solutions like the Emden-curves of the cases treated in II.

The properties of these other solutions can of course be directly derived in detail from the equation (21), but it is instructive to see how far we can proceed with the grid. For this purpose one must know something about what the other solutions do as $t \to \infty$, and it is obvious from (21) that they oscillate about at least one of $v = +1$, 0, -1 and cannot tend to zero, infinity, or a finite limit. We argue here in (θ, ξ) coordinates in which the diagrams are easiest to comprehend.

On referring to Fig. 1, let P_1 be any point below the envelope $\theta = \{\frac{1}{8}(n+1)\}^{\frac{1}{n-1}}\xi^{-\frac{1}{2}}$. Then by using, as in II, the argument that a solution which is not an Emden curve may never touch any one of the family of Emden curves, we can prove that any solution such as $P_1 1$ which starts inwards above the upper Emden curve through P_1 must pass above the envelope. Any solution such as $P_1 2$ which starts inwards between the two Emden curves through P_1 must lie as shown in Fig. 1 and tend steadily to infinity as $\xi \to 0$, always lying below the envelope. Lastly, any solution which starts inwards below the lower Emden curve through P_1 must cut the ξ-axis at some point between 0 and the abscissa of P_1. It cannot cut the Emden curve itself without touching an Emden curve and cannot lie continually between $\xi = 0$ and the Emden curve without making $v(t) \to 0$.

Again, any solution which passes above or starts above the envelope may not stay above the envelope, for $v(t) \geqslant k > 1$ as $t \to \infty$,

and therefore must cut the envelope again, and so the Emden curves, in such a way that it must cut the ξ-axis again as in the last case.

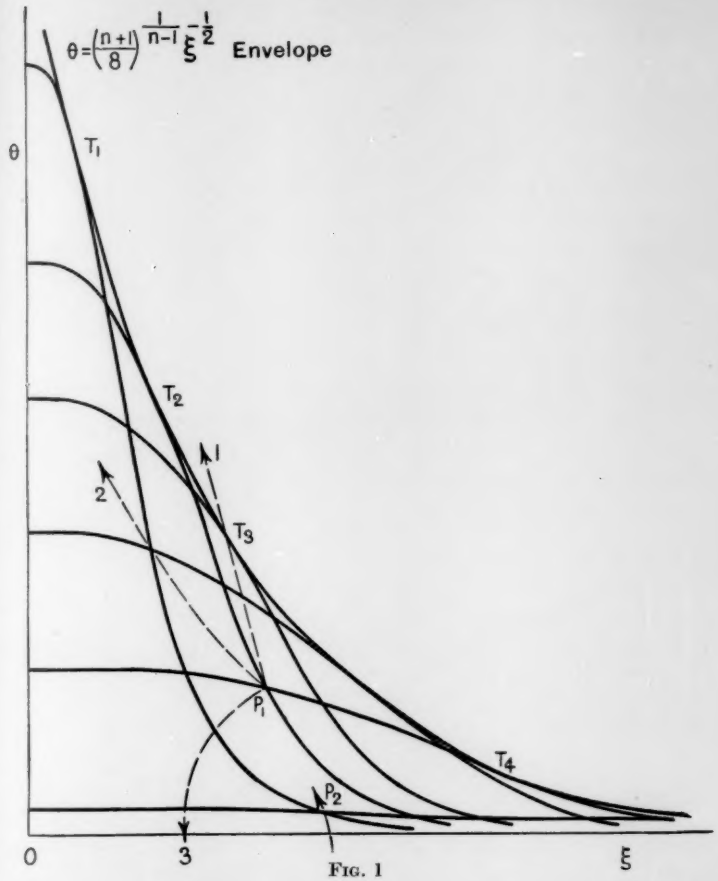

FIG. 1

It can never touch the envelope. Further, solutions which cut the ξ-axis at no matter what angle, whether from above or below, must lie near the axis relative to the Emden curves there as shown at P_2 and must pass above the envelope again. For obviously as P tends to the ξ-axis the slope of the two Emden curves through P both tend

to zero, the Emden curves of this equation having nowhere a zero. We have thus proved:

THEOREM I. *As $\xi \to 0$, solutions of the equation* (A) $(2\sigma+n+3=0)$ *other than Emden's solutions must either* (1) *oscillate infinitely and infinitely often, cutting infinitely often both the curves*

$$\theta = \pm\left(\frac{n+1}{8}\right)^{\frac{1}{n-1}}\xi^{-\frac{1}{2}}, \tag{28}$$

or (2) *tend monotonically to $+\infty$ (or $-\infty$) always lying below (or above) the relevant curve* (28). *The slopes of the Emden curves through any point P between the curves* (28) *enclose between them the slopes of all the solutions which tend monotonically to $\pm\infty$. Solutions through all other points or of any other slope oscillate infinitely as $\xi \to 0$.*

To prove more than this one must of course go back to equation (21). It is then simple to prove by studying the roots of the expression under the square root that the oscillating θ-solutions are solutions in (v,t) which oscillate continually and with constant form about all three of the lines $v=0, \pm 1$. These solutions therefore satisfy the asymptotic form

$$\theta_{\max} = A\xi^{-\frac{1}{2}} \qquad \left(A > \left(\frac{n+1}{8}\right)^{\frac{1}{n-1}}\right), \tag{29}$$

$$\xi_{s+1} = B\xi_s \qquad (B<1), \tag{30}$$

where ξ_{s+1}, ξ_s are any two consecutive zeros. B is a definite function of A but of a form not explicitly determinable. The monotonic θ-solutions are solutions in (v,t) which oscillate continually and with constant form about the line $v=+1$ (or $v=-1$) only. These monotonic (!) solutions therefore satisfy the asymptotic conditions

$$C_1\xi^{-\frac{1}{2}} \leqslant \theta(\xi) \leqslant C_2\xi^{-\frac{1}{2}} \qquad \left(C_1>0,\ C_2<\left(\frac{n+1}{8}\right)^{\frac{1}{n-1}}\right), \tag{31}$$

C_1 and C_2 are functionally related, and $\theta(\xi)$ takes repeatedly all values in the range as $\xi \to 0$. $C_1 \to 0$ when $C_2 \to \left(\frac{n+1}{8}\right)^{\frac{1}{n-1}}$, and $C_1 \to (\frac{1}{4})^{\frac{1}{n-1}}$ when $C_2 \to (\frac{1}{4})^{\frac{1}{n-1}}$. The limiting forms of these solutions are the special solutions $v(t) = \pm 1$ i.e.

$$\theta = \pm(\tfrac{1}{4})^{\frac{1}{n-1}}\xi^{-\frac{1}{2}}. \tag{32}$$

6. Equation (A) ($2\sigma+n+3<0$). We can discuss this equation in the form

$$v''+bv'+c(v^n-v)=0, \tag{33}$$

where $b=2\varpi-1>0$, $c=\varpi(1-\varpi)>0$, and $t\to\infty$. Except for certain details arising from the extra complication of the function v^n-v in place of v^n the discussion follows very closely the discussion of $y''+y'+y^n=0$ in II, and proofs which are mere transcriptions of proofs there given will not be repeated.

Consider first the limited case in which it is assumed that there exists no t_0 such that $v(t)\leqslant 1$ for $t>t_0$. For this case we can prove

THEOREM II. *If there exists no value t_0 such that $|v(t)|\leqslant 1$ when $t>t_0$, any proper solution $v=v(t)$ of equation* (A) ($2\sigma+n+3<0$) *is such that as $t\to\infty$ either $v(t)\to+1$ or $v(t)\to-1$ with or without an infinity of intersections with $v=+1$ or $v=-1$ respectively.*

There are here four *a priori* possibilities of behaviour relative to the line $v=+1$, with corresponding ones for $v=-1$:

(i) There exists a t_0 such that, when $t>t_0$, $v(t)>1+k$, $k>0$.

(ii) For any positive k there exists a t_0 such that, when $t>t_0$, $v(t)\geqslant 1$ and $v=v(t)$ cuts $v=1+k$ in a $\{t_s\}$.*

(iii) $v=v(t)$ oscillates finitely or infinitely about $v=1$ cutting both $v=1+k$ and $v=1$ in $\{t_s\}$'s.

(iv) $v(t)\to 1$ as $t\to\infty$ with or without cutting $v=1$ in a $\{t_s\}$.

The theorem consists in the impossibility of sub-cases (i), (ii), (iii). Sub-cases (i), (ii) are both impossible by proofs as in II which will be omitted. Sub-case (iii) is also impossible, the proof in II requiring some amplifications. Assuming for the moment that between t_s and t_{s+1}, any pair of consecutive intersections of $v=v(t)$ and $v=1$, there is just one value of t, τ_s, at which $v'(\tau_s)=0$, then it follows just as in II that

$$[v'(t_s)]^2>f\{v(\tau_s)\}>[v'(t_{s+1})]^2.$$

The substitution of a more complicated function for $\dfrac{2}{n+1}[v(\tau_s)]^{n+1}$ with the same essential properties makes no difference. It follows further, just as in II, that $|v'(t_s)|\to l\neq 0$ is impossible, and therefore that

$$|v'(t_s)|\to 0, \tag{34}$$

and that (34) implies that

$$v^2(\tau_s)\to 1. \tag{35}$$

* By $\{t_s\}$ we denote an enumerable sequence of values of t tending steadily to infinity.

Let us return next to the uniqueness of τ_s. This presents no difficulty on arcs on which $v(t) \geqslant 1$. On arcs for which $v(t) \leqslant 1$ either of the configurations shown in Figs. 2 and 3 are possible. It is further easy to show, by considering the sign of $v''(t)$ near a zero of $v'(t)$, that

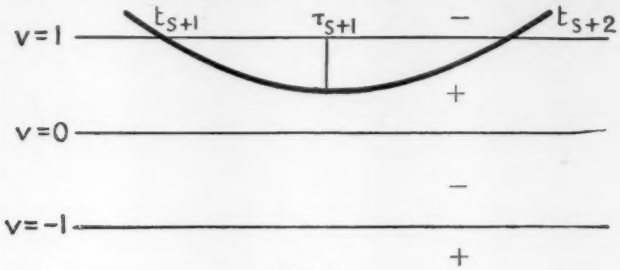

FIG. 2

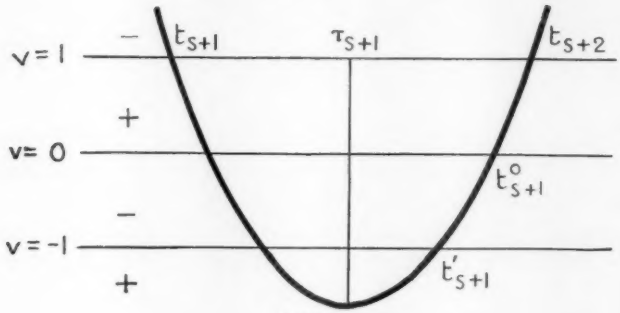

FIG. 3

the only possible way in which $v'(t)$ can vanish more than once between t_{s+1} and t_{s+2} is as shown in Fig. 4. The signs of $v''(t)$ in the different regions are indicated in the figures.

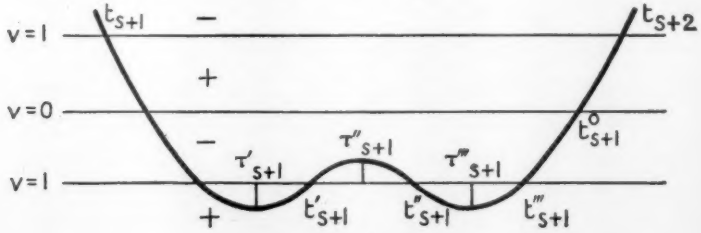

FIG. 4

But then we can conclude by the same argument, which only requires that $v'(t)$ should preserve its sign on the arc used, that

$$[v'(t_{s+1})]^2 > f\{v(\tau'_{s+1})\}, \qquad f\{v(\tau'''_{s+1})\} > [v'(t_{s+2})]^2.$$

But now, on applying the same argument to the intersections of $v = v(t)$ with $v = -1$, we can conclude that

$$f\{v(\tau'_{s+1})\} > [v'(t'_{s+1})]^2 > f\{v(\tau''_{s+1})\} > [v'(t''_{s+1})]^2 > f\{v(\tau'''_{s+1})\},$$

the same function $f(v)$ being involved. The essential conclusions, that

$$|v'(t_{s+1})| > |v'(t_{s+2})|, \qquad v'(t_s) \to 0,$$

remain valid.

Now that we know that $v'(t_s) \to 0$, we can return and re-examine whether arcs such as those shown in Figs. 3, 4 can continue to survive as $t \to \infty$. Suppose they do survive. Consider the arc starting from t'_{s+1} in Fig. 3, on which, for large enough s, $v'(t'_{s+1})$ like $v'(t_s)$ can be made as small as we please. On the arc $t'_{s+1}t$ $(t < t^0_{s+1})$

$$-v'' = bv' + c(v^n - v).$$

Both terms on the right are positive, since $v'(t) > 0$ and $-1 < v(t) < 0$. Therefore

$$-v'' > -c(v - v^n).$$

All along this arc $v = v(t)$ turns over more quickly at corresponding heights than the solution $v = V(t)$ of

$$-v'' = -c(v - v^n),$$

which has the same starting-point and starting-slope. The curve $v = v(t)$ must therefore attain its maximum at which $v'(t) = 0$ nearer* to $v = -1$ than $v = V(t)$. But by direct integration

$$[V'(t'_{s+1})]^2 = 2c\left\{\frac{[V(\tau)]^{n+1}}{n+1} - \frac{[V(\tau)]^2}{2} + \frac{n-1}{2(n+1)}\right\} = f\{V(\tau)\},$$

and the root $V(\tau)$ of this equation nearest to $V = -1$ approaches $V = -1$ as a limit, as $V'(t'_{s+1}) \to 0$. The first vanishing-point of $v'(t)$ must lie even nearer. Hence the forms shown in Figs. 3, 4 are impossible for $t > t_0$, and $v = v(t)$ reduces to a curve such that either $v(\tau_s) \to +1$ or $v(\tau_s) \to -1$. Since, by definition, $v(\tau_s)$ represents the points of maximum excursion from $v = +1$ or $v = -1$, we can conclude that on these solutions either $v(t) \to +1$ or $v(t) \to -1$ as $t \to \infty$.

* It does not appear to be simple to prove this statement rigorously except by a *reductio ad absurdum*, starting from the hypothesis that the zero of $v'(t)$ lies further from $v = -1$ than the zero of $V'(t)$.

A solution may start (as we shall later prove rigorously) by oscillating about all of $v = 0, \pm 1$, but, if it continues ever to lie above $v = +1$ or below $v = -1$ as $t \to \infty$, then $v(t) \to +1$ or $v(t) \to -1$ respectively. This completes the proof of Theorem II.

One further point should be made, and that is that *configurations such as those shown in Fig.* 4 *are everywhere impossible.* This is most simply proved by showing that, to the *left* of an arc such as $t^0_{s+1}t''_{s+1}$, the next arc in the region $-1 < v < 0$ is everywhere steeper at corresponding heights and therefore must again intersect $v = 0$. The proof follows at once from the usual inequalities based on the sign of b and can be left to the reader.

The remaining alternative is that there exists a t_0 such that $|v(t)| \leqslant 1$ for $t > t_0$. To cover this we prove

THEOREM III. *If there exists a* t_0 *such that* $|v(t)| \leqslant 1$, *when* $t > t_0$, *then any proper solution* $v = v(t)$ *of equation* (A) $(2\sigma + n + 3 < 0)$ *is such that, as* $t \to \infty$, *either* $v(t) \to +1 - 0$, $v(t) \to 0$, *or* $v(t) \to -1 + 0$ *monotonically.*

We can observe at once that proper solutions restricted to $|v(t)| \leqslant 1$ cannot cut $v = k$ $(0 \leqslant k < 1)$ twice and lie between $v = k$ and $v = 1$ between the intersections; nor cut $v = -k$ $(0 \leqslant k < 1)$ twice and lie between $v = -k$ and $v = -1$ between the intersections. It follows at once that for any given k $(-1 < k < 1)$ there exists a t_0 such that, when $t > t_0$, either $v(t) > k$ or $v(t) < k$, and therefore that there must exist an l $(-1 \leqslant l \leqslant 1)$ such that $v(t) \to l$ monotonically. Proper solutions for which $l = 0$ will be found to exist (Emden's solutions). Proper solutions for which $l = \pm 1$ may also exist, but they are, in no essential, different from those allowed by Theorem I. Proper solutions for other values of l are impossible. For then

$$v'' + bv' \sim \alpha \, (\neq 0), \qquad v' + bv \sim \alpha t,$$

$$ve^{bt} \sim \frac{\alpha}{b} te^{bt}, \qquad v \sim \frac{\alpha}{b} t,$$

which contradicts $v(t) \to l$. This completes the proof of Theorem III.

When $v(t) \to 0$ monotonically the form of the solutions, if they exist, can be determined at once by Hardy's theorem.* For, on writing $v' = \eta$, the equation becomes

$$\eta \frac{d\eta}{dv} + b\eta - c(v - v^n) = 0, \tag{36}$$

* Hardy, *Proc. London Math. Soc.*, vol. 10 (1911), p. 451. See also II, p. 69.

and the form of solutions is required for which $v \to 0$, $\eta \to 0$, $t \to \infty$. It is easily found that such solutions must be of the form

$$v(t) \sim Ae^{-\varpi t},$$

where A is a constant of integration. Such solutions satisfy also

$$\theta(\xi) \to C.$$

They are Emden's solutions, and their actual existence and uniqueness may be established as in II. We embody these results in

THEOREM IV. *Emden's solutions of equation* (A) $(2\sigma+n+3<0)$ *satisfying*

$$\theta(\xi) \to C \qquad (\xi \to 0)$$

where C is arbitrary, exist and are unique. If $v(t) \to 0$ as $t \to \infty$ on any proper solution, then must

$$v(t) \sim Ae^{-\varpi t}$$

and the solution must be an Emden solution.

A more exact study of the solutions $v(t) \to \pm 1$ is also possible and will be given later. It is best to discuss first equation (A) when $2\varpi-1<0$, $t \to \infty$, for, since it is then the same equation as when $2\varpi-1>0$, $t \to -\infty$, knowledge of one case can be used to help with the other.

7. Equation (A) $(2\sigma+n+3>0)$. The form of the equation is now

$$v''-bv'+c(v^n-v)=0, \tag{37}$$

where $b=-(2\varpi-1)>0$, $c=\varpi(1-\varpi)>0$, and $t \to \infty$. We prove first

THEOREM V. *Equation* (A) $(2\sigma+n+3>0)$ *possesses no proper solutions, other than $v(t)=\pm 1$, such that either $v(t) \geqslant 1$ when $t>t_0$, or $v(t) \leqslant -1$ when $t>t_0$, or $v(t) \to 1-0$, or $v(t) \to -1+0$ monotonically.*

Solutions obeying the restriction $v(t) \geqslant 1$ must be ultimately monotonic and their asymptotic behaviour can therefore be studied by Hardy's theorem applied to

$$\eta\frac{d\eta}{dv}-b\eta+c(v^n-v)=0. \tag{38}$$

It is easily shown by examination of each alternative that $v(t) \to \infty$, $v(t) \to l$, and $v(t) \to 1+0$ all lead to contradictions. The details are left to the reader. We can also conclude by the same argument that $v(t) \to 1-0$ monotonically is equally impossible and also the alternatives concerning $v=-1$. Hence the theorem.

We prove next

THEOREM VI. *The only proper solutions, other than* $v(t) = \pm 1$, *of equation* (A) $(2\sigma+n+3>0)$,

which satisfy the restriction that

$$-1 \leqslant v(t) \leqslant 1$$

when $t > t_0$, *are Emden's solutions which exist, are unique, and satisfy*

$$v(t) \sim Ae^{-\omega t}, \qquad \theta(\xi) \to C \qquad (\xi \to 0).$$

We know already that $v(t) \to +1, -1$ are impossible. Two consecutive zeros of $v'(t)$, if they exist, must lie on opposite sides of

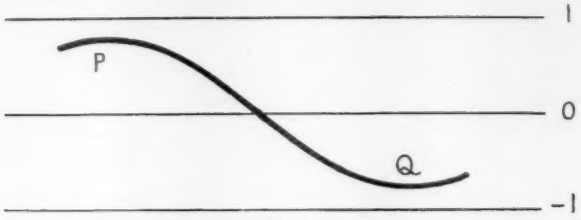

FIG. 5

$v = 0$, and therefore must be arranged as in Fig. 5. But $v'' > 0$ at P and $v'' < 0$ at Q, so that this is impossible. Hence all such $v = v(t)$ are ultimately monotonic and therefore $v(t) \to l$ $(-1 < l < 1)$. If $v(t) \to l\,(\neq 0)$, then

$$v'' - bv' \sim \alpha\,(\neq 0), \qquad v' - bv \sim \alpha t,$$

$$\frac{d}{dt}(ve^{-bt}) \sim \alpha t e^{-bt}, \qquad ve^{-bt} - C_1 \sim C_2 - \frac{\alpha t}{b}e^{-bt},$$

$$v(t) \to \infty,$$

which contradicts $v(t) \to l$.

The only alternative is $v(t) \to 0$ monotonically. Just as before, we can prove that Emden's solutions exist and are unique, and that, when $v(t) \to 0$, $v = v(t)$ must be one of Emden's solutions. This completes the proof of Theorem VI.

The remaining possibility is that $v = v(t)$ cuts $v = 1$ and (or) $v = -1$ infinitely often in a $\{t_s\}$. In this case we shall prove that $v = v(t)$ must ultimately oscillate infinitely and infinitely often, cutting all three of $v = 0, \pm 1$ in $\{t_s\}$'s. Combining this result with the results of Theorems V and VI, we thus prove

THEOREM VII. *If a proper solution of equation* (A) *is not one of Emden's solutions or one of the special solutions* $v(t) = \pm 1$, *then, as* $t \to \infty$, $v = v(t)$ *oscillates infinitely and infinitely often, cutting* $v = 0, \pm 1$ *in* $\{t_s\}$*'s.*

The arguments to be used are exactly the same as those of the corresponding part of § 6, except that the sign of b is changed so that the sense of every inequality is reversed. We can therefore prove at once that, if $\{t_s\}$ is the sequence of intersections of $v = v(t)$ with $v = 1$, say, then

$$|v'(t_s)| < |v'(t_{s+1})|,$$

and further that, as $t_s \to \infty$,

$$|v'(t_s)| \to \infty.$$

We have already proved in § 6 that oscillations about $v = +1$ (or $v = -1$) and $v = 0, \pm 1$ cannot alternate; the same argument shows that any oscillation which succeeds an oscillation about $v = 0, \pm 1$ as $t \to \infty$ is also an oscillation about $v = 0, \pm 1$. It therefore only remains to show that since $|v'(t_s)| \to \infty$ it is impossible for the oscillations to remain confined to $v = +1$ (or $v = -1$). Consider an oscillation below $v = +1$ starting with a given sufficiently large value of $|v'(t_s)|$. On this arc

$$v'' = bv' + c(v - v^n)$$

and $bv' < 0$, so that $\qquad v'' < c(v - v^n)$;

the negative slope diminishes less rapidly than on the curve $v = V(t)$ which is the solution of

$$v'' = c(v - v^n)$$

with the same starting-point and starting-slope on $v = 1$. Therefore at corresponding heights on these arcs

$$|v'(t)| > |V'(t)|.$$

But by direct integration we can show that $v = V(t)$ must intersect $v = 0$ when $|v'(t_s)|$ is sufficiently large. Therefore $v = v(t)$ must also intersect $v = 0$, and therefore $v = 0, \pm 1$, for sufficiently large $|v'(t_s)|$ and therefore when $t > t_0$. This completes the proof of Theorem VII.

8. Properties of the Emden solutions. Consider first the case $2\sigma + n + 3 < 0$. Then the Emden solutions in (v, t) coordinates are those solutions which satisfy $v(t) \sim Ae^{-\omega t}$ as $t \to \infty$. They are obviously all the same curve with variable origins for t. We require to

know how they behave as we follow them backwards towards $t \to -\infty$. In this direction they are controlled by Theorems V–VII and therefore, as $t \to -\infty$, $v = v_E(t)$ either satisfies $v_E(t) \to 0$ and is again an Emden solution, or else oscillates continually about $v = 0, \pm 1$. In order to turn over so that $v_E(t)$ can start diminishing as we trace $v = v_E(t)$ backwards it is necessary for $v = v_E(t)$ to rise above $v = 1$. [It is tacitly assumed that $v_E(t)$ is positive for large t.] Now it is impossible for $v = v_E(t)$ to cut $v = 1$ more than twice, so making a loop below it before it cuts $v = 0$. For on the arc which starts off below $v = 1$ the slope must be everywhere steeper at equal heights than on the Emden branch $t \to +\infty$, and this is impossible if $v'_E(t)$ vanishes for positive v. Thus the only remaining possibilities are those shown in Fig. 6. And in fact it is easily seen that the dotted alternative is impossible. For in that case, on the arc t_0, $+\infty$,

$$v'' = -bv' + c(v - v^n),$$

$$v'' > c(v - v^n),$$

$$-v'v'' > -c(v - v^n)v',$$

$$\tfrac{1}{2}[v'(t_0)]^2 > c \int_0^1 (v - v^n)\, dv.$$

Similarly on the arc $-\infty t_1$

$$\tfrac{1}{2}[v'(t_1)]^2 < c \int_0^1 (v - v^n)\, dv.$$

But from the arc $t_1 t_0$ $\qquad |v'(t_0)| < |v'(t_1)|$,

and these are contradictory. Hence the case of the dotted curve is impossible and the full curve of Fig. 6 shows the nature of the Emden solutions. We observe that, between the last zero z and ∞, $v'(t)$ vanishes just once, and therefore that any two curves of the family of Emden curves stopped off at their first zero intersect once only, and that the family has a single envelope $v = \alpha > 1$. In this case therefore the family of Emden curves still forms a grid which can be used to define the behaviour of the other solutions.

When $2\sigma + n + 3 > 0$ similar arguments may be applied. The Emden curve $v = v_E(t)$, when continued back towards $t \to -\infty$, must either be again an Emden curve or tend to $v = +1$ or $v = -1$ with or without oscillations. The argument just given shows that it cannot be an Emden curve again. [This happens only when $2\sigma + n + 3 = 0$.]

It cannot approach $v = -1$ without cutting $v = 0$, and such an arc is impossible because its slope must be less than the slope of the

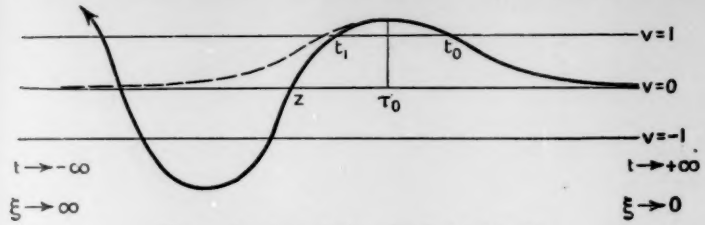

FIG. 6. Emden solutions for $2\sigma+n+3<0$.

Emden arc at corresponding heights. Its nature is therefore fully shown in Fig. 7.

Emden's solutions never vanish. It is at once evident that such curves may have an infinity of envelopes and any pair an infinity of intersections before they reach $v = \theta = 0$, so that they no longer have the properties necessary for use as a grid. And in fact the other solutions, except the special ones $v = \pm 1$, are now all of one type as $t \to +\infty$.

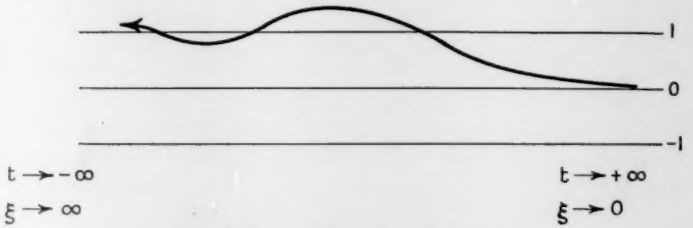

FIG. 7. Emden solutions for $2\sigma+n+3>0$.

We can now use the grid of Emden solutions, but only when $2\sigma+n+3<0$, just as we have already used it for $2\sigma+n+3=0$ and for $\sigma+n+1\leqslant 0$, $\sigma+2<0$ in II. As a result we can establish the final points required for the following theorems, which also recapitulate information given previously.

THEOREM VIII. *If $n = p/q$, where p, q are odd and if*

$$\sigma+n+1>0, \qquad 2\sigma+n+3<0, \qquad \sigma+2<0,$$

then no proper solution of equation (A)

$$\theta''+x^{\sigma}\theta^{n}=0 \qquad (x \to \infty)$$

or of the equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\xi^{-\sigma-4}\theta^n=0 \qquad (\xi\to 0)$$

can continue to oscillate about $\theta=0$.

Such solutions must take one or other of the asymptotic forms

$$\theta=C-\frac{\alpha C^n x^{\sigma+2}}{(\sigma+1)(\sigma+2)} \qquad (\alpha\sim 1), \qquad [E\text{-solutions}]$$

where C *is a constant of integration, or*

$$\theta\sim\pm\left[\frac{-(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}}x^{-\frac{\sigma+2}{n-1}}, \qquad [M\text{-solutions}]$$

of which the former are exceptional. There are two special M*-solutions,*

$$\theta=\pm\left[\frac{-(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}}x^{-\frac{\sigma+2}{n-1}}.$$

THEOREM IX. *Through any point* P, θ_1, ξ_1 $(\theta_1>0)$ *below the envelope of the* E*-solutions of equation* (A) $(2\sigma+n+3<0)$ *namely*

$$\theta=m/\xi^{\frac{\sigma+2}{n-1}} \qquad \left(m>\left[\frac{-(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}}\right),$$

pass two E*-solutions with slopes** α, β, $\alpha>\beta$. (i) *If* $\alpha>\theta'(\xi_1)>\beta$, $\theta=\theta(\xi)$ *is an* M*-solution, monotonic when* $\xi<\xi_1$, *and* $\theta(\xi)\to+\infty$ *as* $\xi\to 0$. (ii) *If* $\theta'(\xi_1)=\alpha$, *or* $\theta'(\xi_1)=\beta$, *the solution is an* E*-solution.* (iii) *If* $\theta'(\xi_1)>\alpha$, *or* $\theta'(\xi_1)<\beta$, *the solution cuts* $\theta=0$ *again at a point* ξ_2 *such that* $0<\xi_2<\xi_1$.

Any proper solution through a point P, θ_1, ξ_1 *above or on the envelope cuts* $\theta=0$ *again at* ξ_2 *where* $0<\xi_2<\xi_1$, *except that through a point* P *on the envelope there passes the single* E*-solution which touches the envelope there.*

Similar relations hold for negative θ_1.

9. More exact asymptotic forms when $2\sigma+n+3<0$. Let us consider more closely the solutions for which $v(t)\to+1$ in this case. Writing $v=1+v$ the equation becomes

$$v''+bv'+(n-1)cv[1+f(v)]=0, \qquad (39)$$

* In the enunciation of this theorem slopes are reckoned positive when $\theta(\xi)$ increases as $\xi\to 0$.

where $f(v)$ is a rational function of v and v^n such that $f(v) \to 0$ as $v \to 0$. It is almost obvious, since we know already that $v(t) \to 0$ on the solutions in which we are interested, that the asymptotic forms of these solutions must be the solutions of the equation

$$v''+bv'+(n-1)cv=0. \tag{40}$$

A rigorous formal proof, however, seems to require the methods used in I for discussing standard form II. The arguments there used can be extended at once to cover what is required here, and we may therefore proceed to use the solutions of (40) as the required asymptotic forms.*

Solutions of (40) are of the form

$$v = A_1 e^{\lambda_1 t} + B_1 e^{\lambda_2 t}$$

where

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4(n-1)c}}{2}. \tag{41}$$

Thus $v=v(t)$ is ultimately monotonic if $b^2 \geqslant 4(n-1)c$ or

$$(2\sigma+n+3)^2 \geqslant -4(n-1)(\sigma+2)(\sigma+n+1). \tag{42}$$

In this case

$$v(t)-1=v(t)=e^{\frac{1}{2}(2\sigma+n+3)t}\left[e^{(\mu+\epsilon_1)t}+e^{-(\mu+\epsilon_2)t}\right] \qquad (\epsilon_1, \epsilon_2 \to 0), \tag{43}$$

where

$$\mu = \frac{\left[(2\sigma+n+3)^2+4(n-1)(\sigma+2)(\sigma+n+1)\right]^{\frac{1}{2}}}{2(n-1)}, \tag{44}$$

and the second term in (43) may only be retained if the first vanishes identically. If, however,

$$(2\sigma+n+3)^2 < -4(n-1)(\sigma+2)(\sigma+n+1), \tag{45}$$

then $v=v(t)$ oscillates continually about $v=0$. If t_s, t_{s+1} are consecutive zeros and τ_s the abscissa of the maximum, then

$$v(\tau_s)-1=v(\tau_s) \sim Ce^{\frac{1}{2}(2\sigma+n+3)\tau_s} \tag{46}$$

where C is a constant of integration, and

$$t_{s+1}-t_s \sim \frac{4\pi(n-1)}{\left[-4(n-1)(\sigma+2)(\sigma+n+1)-(2\sigma+n+3)^2\right]^{\frac{1}{2}}}. \tag{47}$$

* An alternative line of argument is to construct by an iterative process exact solutions of equation (39) of the required form by using the solutions of (40) as starting-points and showing that these solutions of (40) are then necessarily the asymptotic forms required for the solutions of (39). This is straightforward and contains no point of novelty.

10. More exact asymptotic forms when $2\sigma+n+3>0$. The necessary theorems to detail to us the forms of the infinite oscillations of the solutions other than Emden's and the special solutions in this case have already been given implicitly in I. Theorem VI (a) of I can be repeated for equation (A) in the following form:

If $\{x_s\}$ *be the sequence of zeros and* X_s, $\theta(X_s)$ *the sequence of maxima and minima on any oscillating solution, and if* $x_{s+1}/x_s \to 1$, *then*

$$|\theta(X_s)| = X_s^{-\frac{\sigma}{n+3}+\epsilon} \qquad (\epsilon \to 0), \tag{48}$$

$$x_{s+1}-x_s = x_s^{-\frac{2\sigma}{n+3}+\epsilon} \qquad (\epsilon \to 0). \tag{49}$$

It is not, however, proved in I that if $\sigma+2<0$, $2\sigma+n+3>0$, then $x_{s+1}/x_s \to 1$.

To supply the missing step it seems to be simplest to extend the results for the standard form II of I, Theorem VI (b). This theorem states *inter alia* that

If $\{t_s\}$ *be the sequence of zeros and* τ_s, $v(\tau_s)$ *the sequence of maxima and minima on any oscillating solution of*

$$v''-bv'+cv^n = 0, \tag{50}$$

and if $t_{s+1}-t_s \to 0$, *then*

$$v(\tau_s) = \exp\left\{\left(\frac{2b}{n+3}+\epsilon\right)t\right\} \qquad (\epsilon \to 0), \tag{51}$$

$$t_{s+1}-t_s = \exp\left\{\left(-\frac{n-1}{n+3}b+\epsilon\right)t\right\} \qquad (\epsilon \to 0). \tag{52}$$

In particular, if as here $b>0$, *then* $t_{s+1}-t_s \to 0$ *and relations* (51), (52) *hold good.*

It will be observed that (50) is not the desired equation and we must repeat the arguments with v^n-v in place of v^n, but we know, to start with, that $v=v(t)$ oscillates about all three of $v=0, \pm 1$, having maxima only above $v=1$ and minima below $v=-1$, and that $|v(\tau_s)| \to \infty$. We can confine attention to $b>0$, $c>0$.

The correct v-equation may be written in the form

$$-\frac{d}{dt}(e^{-bt}v')^2 = 2ce^{-2bt}(v^n-v)v'. \tag{53}$$

If t_0 be any point on the arc $t_s\tau_s$, then for $t_0 \leqslant t \leqslant \tau_s$

$$e^{-2b\tau_s} \leqslant e^{-2bt} \leqslant e^{-2bt_0},$$

and therefore

$$2ce^{-2b\tau_s}(v^n-v)v' \leqslant -\frac{d}{dt}(e^{-bt}v')^2 \leqslant 2ce^{-2bt_0}(v^n-v)v'.$$

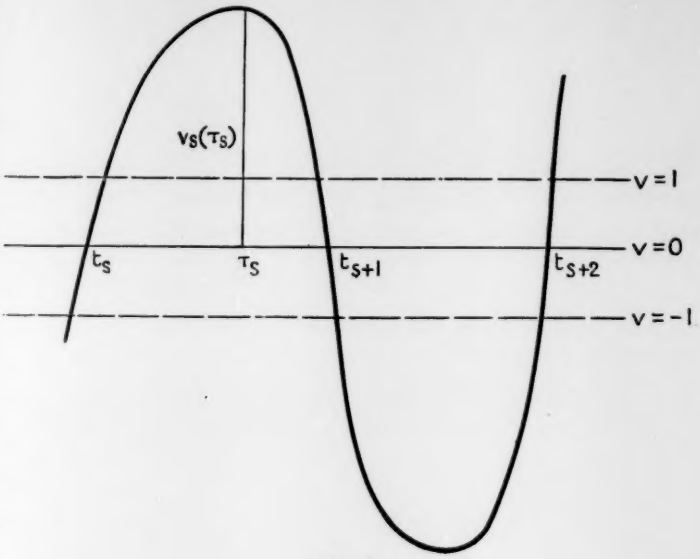

FIG. 8

Integrating this over the range $t_0\tau_s$ we obtain

$$2ce^{-2b\tau_s}\left\{\frac{[v(\tau_s)]^{n+1}}{n+1}-\frac{[v(\tau_s)]^2}{2}-\frac{[v(t_0)]^{n+1}}{n+1}+\frac{[v(t_0)]^2}{2}\right\}$$

$$< e^{-2bt_0}[v'(t_0)]^2 < 2ce^{-2bt_0}\left\{\frac{[v(\tau_s)]^{n+1}}{n+1}-\frac{[v(\tau_s)]^2}{2}-\frac{[v(t_0)]^{n+1}}{n+1}+\frac{[v(t_0)]^2}{2}\right\}.$$

These inequalities hold for all values of t_0 in the interval and therefore

$$2ce^{-2b(\tau_s-t)}\left\{\frac{[v(\tau_s)]^{n+1}}{n+1}-\frac{[v(\tau_s)]^2}{2}-\frac{v^{n+1}}{n+1}+\frac{v^2}{2}\right\}$$

$$< (v')^2 < 2c\left\{\frac{[v(\tau_s)]^{n+1}}{n+1}-\frac{[v(\tau_s)]^2}{2}-\frac{v^{n+1}}{n+1}+\frac{v^2}{2}\right\}. \qquad (54)$$

Let us now compare the $v = v(t)$ curve with the curve

$$v = f_Q(\psi), \qquad (55)$$

where $f_Q(\psi)$ is the perfectly definite real function defined by

$$\left\{\frac{Q^{n+1}}{n+1}-\frac{Q^2}{2}-\frac{f_Q^{n+1}}{n+1}+\frac{f_Q^2}{2}\right\}^{-\frac{1}{2}}df_Q=d\psi \tag{56}$$

taken with a suitable origin. Let

$$\int_0^Q\left\{\frac{Q^{n+1}}{n+1}-\frac{Q^2}{2}-\frac{f_Q^{n+1}}{n+1}+\frac{f_Q^2}{2}\right\}^{-\frac{1}{2}}df_Q=K(Q); \tag{57}$$

then $f_Q\{K(Q)\}=Q$. As t goes from t_s to τ_s, ψ goes from 0 to $K(Q)$ and $v(t)$ goes from 0 to $v(\tau_s)$, with a similar relationship for the second part of the wave. On comparing the inequalities (54) with (55) and (56), we obtain

$$2ce^{-2b(\tau_s-t)}<(\psi')^2<2c;$$

we take the square root of these inequalities (all square roots being positive) and integrate from t_s to τ_s, thus obtaining

$$\frac{(2c)^{\frac{1}{2}}}{b}\{1-e^{-b(\tau_s-t_s)}\}<K(Q)<(2c)^{\frac{1}{2}}(\tau_s-t_s). \tag{58}$$

Similar arguments applied to the arc $\tau_s t_{s+1}$ show that

$$(2c)^{\frac{1}{2}}(t_{s+1}-\tau_s)<K(Q)<\frac{(2c)^{\frac{1}{2}}}{b}\{e^{b(t_{s+1}-\tau_s)}-1\}. \tag{59}$$

It is immediately obvious that we have only to prove that $K(Q)\to 0$ as $Q(=|v(\tau_s)|)\to\infty$, in order to deduce that $t_{s+1}-t_s\to 0$ and so complete the investigation of this section. By (57)

$$K(Q)=\frac{(n+1)^{\frac{1}{2}}}{Q^{\frac{1}{2}(n-1)}}\int_0^1\left\{1-x^{n+1}-\frac{n+1}{2}\,\frac{1}{Q^{n-1}}(1-x^2)\right\}^{-\frac{1}{2}}dx,$$

$$=\frac{(n+1)^{\frac{1}{2}}}{Q^{\frac{1}{2}(n-1)}}\int_0^1(1-x^{n+1})^{-\frac{1}{2}}\left\{1-\frac{n+1}{2}\,\frac{1}{Q^{n-1}}\,\frac{1-x^2}{1-x^{n+1}}\right\}^{-\frac{1}{2}}dx.$$

Over the range $0\leqslant x\leqslant 1$ the coefficient of $1/Q^{n-1}$ in the second factor in the integrand is bounded, and therefore, as $Q\to\infty$,

$$K(Q)\sim\frac{(n+1)^{\frac{1}{2}}}{Q^{\frac{1}{2}(n-1)}}\int_0^1(1-x^{n+1})^{-\frac{1}{2}}\,dx\to 0.$$

These results may be embodied in the following:

THEOREM X. *If as $x \to \infty$ $\{x_s\}$ be the sequence of zeros and X_s, $\theta(X_s)$ the sequence of maxima and minima on any oscillating solution of equation* (A) $(2\sigma+n+3>0)$, *then*

$$|\theta(X_s)| = X_s^{-\frac{\sigma}{n+3}+\epsilon} \qquad (\epsilon \to 0), \tag{48}$$

$$x_{s+1}-x_s = x_s^{-\frac{2\sigma}{n+3}+\epsilon} \qquad (\epsilon \to 0). \tag{49}$$

*These expressions may also be obtained in the more exact forms**

$$|\theta(X_s)| \sim A X_s^{-\frac{\sigma}{n+3}}, \tag{60}$$

$$x_{s+1}-x_s \sim \frac{1}{A^{\frac{1}{2}(n-1)}}\sqrt{\left(\frac{2}{n+1}\right)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n+1}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n+1}\right)} X_s^{-\frac{2\sigma}{n+3}}. \tag{61}$$

11. Tabulations of the behaviour of the proper solutions of the equation

$$\theta''+x^\sigma\theta^n = 0 \qquad (n>1,\ n=p/q,\ p,\ q \text{ odd}). \tag{2}$$

We have now a fairly complete knowledge of the behaviour of any proper solution of equation (2) which may be exhibited as follows:

Table I

The form as $x \to \infty$ of proper solutions of

$$\theta''+x^\sigma\theta^n = 0 \qquad (n>1,\ n=p/q,\ p,\ q \text{ odd}).$$

Case I.† $\sigma+2 \geqslant 0$, $2\sigma+n+3>0$, $\sigma+n+1>0$.

All solutions must oscillate about $\theta=0$ with the asymptotic forms

$$\left.\begin{aligned} |\theta(X_s)| &\sim A X_s^{-\frac{\sigma}{n+3}}, \\ x_{s+1}-x_s &\sim \frac{1}{A^{\frac{1}{2}(n-1)}}\left(\frac{2}{n+1}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n+1}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n+1}\right)} X_s^{-\frac{2\sigma}{n+3}}, \end{aligned}\right\} \tag{P}$$

where A is a constant of integration, $\{X_s\}$ the sequence of stationary points of $\theta(x)$, and $\{x_s\}$ its sequence of zeros.

* By the methods outlined in I, §§ 12–20.

† Sources: $\sigma+2>0$, I, Table III; $\sigma+2=0$, I, Table V, with a mistake corrected, which occurred in transcribing the correct results of I, Theorem VI (*b*).

Case II. $\sigma+2<0$, $2\sigma+n+3>0$, $\sigma+n+1>0$.

There are solutions of three distinct types, some oscillating, some ultimately monotonic.

(i) *Special solutions (with no arbitrary constants)*

$$\theta = \pm\left[\frac{-(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}} x^{-\frac{\sigma+2}{n-1}}. \tag{Q}$$

(ii) *Emden's solutions (with one arbitrary constant C)*

$$\theta = C - \frac{\alpha C^n x^{\sigma+2}}{(\sigma+1)(\sigma+2)} \qquad (\alpha \sim 1). \tag{R}$$

Emden's solutions never vanish for any value of x, and are everywhere monotonic.

(iii) *All other solutions (with two arbitrary constants) must oscillate, having oscillations of form* (P), *Case I.*

Case III. $\sigma+2<0$, $2\sigma+n+3=0$, $\sigma+n+1>0$.

There are solutions of four distinct types, some oscillating, some ultimately monotonic.

(i) *Special solutions of form* (Q), *Case II.*

(ii) *Emden's solutions (with one arbitrary constant C)*

$$\theta = \pm\left[\frac{\frac{1}{2}(n+1)C^2}{\{1+C^2/x^{\frac{1}{2}(n-1)}\}^2}\right]^{\frac{1}{n-1}}. \tag{S}$$

The exact form (S) *conforms to* (R), *Case II.*

(iii) *Monotonic solutions (with two arbitrary constants) having no zero which tend to $+\infty(-\infty)$. These solutions $(+\infty)$ satisfy the relations*

$$\left.\begin{aligned} C_1 x^{\frac{1}{2}} \leqslant \theta(x) \leqslant C_2 x^{\frac{1}{2}}, \\ C_1>0, \qquad C_2<\left(\frac{n+1}{8}\right)^{\frac{1}{n-1}}, \qquad C_1=f(C_2), \end{aligned}\right\} \tag{T}$$

and $\theta(x)$ takes all values repeatedly in this range as $x\to\infty$.

$$C_1\to 0, \text{ as } C_2\to\left(\frac{n+1}{8}\right)^{\frac{1}{n-1}};\ C_1\to\left(\tfrac{1}{4}\right)^{\frac{1}{n-1}}, \text{ as } C_2\to\left(\tfrac{1}{4}\right)^{\frac{1}{n-1}}.$$

(iv) *Solutions (with two arbitrary constants) which continue to oscillate about zero as $x\to\infty$, of the form*

$$\left.\begin{aligned} |\theta(X_s)| = AX_s^{\frac{1}{2}}, \qquad A>\left(\frac{n+1}{8}\right)^{\frac{1}{n-1}}, \\ x_{s+1}=Bx_s, \qquad B>1, \qquad B=g(A). \end{aligned}\right\} \tag{U}$$

All these relations in Case III are exact, not asymptotic.

Emden's solutions form a grid separating the monotonic from the oscillating solutions.

Case IV. $\sigma+2<0$, $2\sigma+n+3<0$, $\sigma+n+1>0$.

All the solutions are ultimately monotonic as $x\to\infty$. *They are of three types.*

(i) *Special solutions of form* (Q), *Case II.*

(ii) *Emden's solutions (with one arbitrary constant) of form* (R), *Case II, but here these solutions have an infinity of zeros as* $x\to 0$.

(iii) *Ultimately monotonic solutions (two arbitrary constants) which satisfy*

$$\theta(x)\sim\pm\left[\frac{-(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}}x^{-\frac{\sigma+2}{n-1}}. \tag{V}$$

These solutions have an infinity of zeros as $x\to 0$.

Emden's solutions form a grid separating off the solutions which have had their last zero as $x\to\infty$ *and become monotonic, from those which have more zeros to come.*

Case V. $\sigma+2<0$, $2\sigma+n+3<0$, $\sigma+n+1=0$.

All the solutions are ultimately monotonic as $x\to\infty$. *They are of only two types.*

(i) *Emden's solutions (with one arbitrary constant) of form* (R), *having zeros as* $x\to 0$, *as in Case IV.*

(ii) *All other solutions (with two arbitrary constants) have the asymptotic form as* $x\to\infty$

$$\theta(x)\sim\pm x\left[\frac{1}{(n-1)\log(Dx)}\right]^{\frac{1}{n-1}}, \tag{W}$$

where D is one of the constants of integration. These solutions all have an infinity of zeros as $x\to 0$.

Emden's solutions form a grid as in Case IV.

Case VI. $\sigma+2<0$, $2\sigma+n+3<0$, $\sigma+n+1<0$.

The types and arrangement of the solutions are as in Case V except that form (W) *is replaced by*

$$\theta(x)\sim Dx, \tag{X}$$

where D is one of the constants of integration.

12. Tabulations of the behaviour of the proper solutions of the equation

$$\theta''+x^{\sigma}\theta^{n}=0 \qquad (n>1,\ n=p/q,\ \text{one of } p,\ q \text{ even}). \tag{2}$$

When n is not odd the only possible change in the arguments concerns the behaviour of a solution which crosses or attempts to cross

the axis $\theta = 0$. When $n = p/q$, p even, the solutions can cross the axis, but on putting $\theta = -\phi$ ($\phi > 0$) the solutions on the other side of the axis satisfy the equation

$$\phi'' - x^\sigma \phi^n = 0, \tag{8}$$

and can be best considered under that heading. When $n = p/q$, q even, proper solutions cannot cross the axis $\theta = 0$. We can therefore agree to study the proper solutions of equation (2) with the extra restriction that they ultimately remain positive. It has of course already been proved in I that no proper solution of (2) for these values of n or of (8) for any value of n can oscillate continually as $x \to \infty$, so that proper solutions $\theta = \theta(x)$ must be ultimately monotonic and also possess a monotonic $\theta'(x)$. The results naturally are that positive proper solutions of (2) in this case must have one of the forms available for monotonic solutions for p, q odd. It is perhaps worth while to repeat the relevant parts of Table I.

Table II

The form as $x \to \infty$ of positive proper solutions of

$$\theta'' + x^\sigma \theta^n = 0 \qquad (n > 1,\ n = p/q,\ p \text{ or } q \text{ even}).$$

Case I. $\sigma + 2 \geqslant 0$, $2\sigma + n + 3 > 0$, $\sigma + n + 1 > 0$.

There are no proper solutions remaining positive as $x \to \infty$. All solutions again cross or attempt to cross $\theta = 0$.

Case II. $\sigma + 2 < 0$, $2\sigma + n + 3 > 0$, $\sigma + n + 1 > 0$.

There are positive proper solutions of the forms (Q) (*special*) *and* (R) (*Emden's*), *but all others again cross or attempt to cross $\theta = 0$.*

Case III. There are positive proper solutions of the forms (Q) (*special*), (S) (*Emden's*), *and* (T) (*with two arbitrary constants*).

Emden's solutions form a grid separating the positive monotonic solutions from the others which must try to have another zero.

Cases IV–VI. The forms of the solutions are in each case identical with the positive forms in Table I. Emden's solutions form a grid separating the positive monotonic solutions from the others which must have another zero.

13. Positive proper solutions of the equation

$$\theta'' - x^\sigma \theta^n = 0 \qquad (n > 1). \tag{8}$$

It has already been shown in I, by arguments that transfer immediately to rational n, that the equation (8) cannot have any solution oscillating about $\theta = 0$. We can therefore confine attention to

the region $\theta \geqslant 0$ and need not distinguish between the various forms of n. The solutions are ultimately monotonic and therefore $\theta(x) \to 0$, l or ∞. Further, $\theta'(x)$ is also monotonic, for it cannot vanish more than once unless $\theta''(x)$, and therefore $\theta(x)$ vanishes between each pair of roots. Thus also $\theta' \to 0$, l, or ∞.

The further analysis in I of the monotonic asymptotic forms is rather incomplete, and is based on certain general theorems which are not immediately applicable for rational n, though the analogous theorems most probably continue to be true. The discussion may, however, be conducted throughout quite simply by the use of Hardy's theorem* and the usual simple arguments. We shall therefore give here a condensed statement of a complete discussion carried out in this way.

Case I. $\sigma+2>0$, $2\sigma+n+3>0$, $\sigma+n+1>0$.

Here $\theta = [\varpi(\varpi-1)]^{\frac{1}{n-1}} x^{\varpi} v, \qquad \varpi = -\dfrac{\sigma+2}{n-1} < 0, \qquad v > 0,$

$$v''-bv'+c(v-v^n)=0 \qquad (b>0,\ c>0), \tag{62}$$

the independent variable being $t = \log x$. The $v'(t)$ of a proper solution of (62) can only continue to have zeros as $t \to \infty$, if $v''(t)$ has a series of zeros and therefore if $v = v(t)$ oscillates about $v = 1$. But this is impossible from the sign of $v''(t)$ near zeros of $v'(t)$. Therefore $v = v(t)$ is ultimately monotonic and $v(t) \to 0$, 1, l $(l \neq 0, 1)$ or ∞. It is easily proved in the manner of § 6 that $v(t) \to 0$, l $(l \neq 0, 1)$ are impossible; $v(t) \to 1$ or $v(t) = 1$ is possible. By writing $v' = \eta$ equation (62) becomes

$$\eta \frac{d\eta}{dv} - b\eta + c(v-v^n) = 0,$$

which can be studied by Hardy's theorem as $v \to \infty$. It is easily found that the only possible form in which $v \to \infty$ is

$$v(t) \sim \left[\frac{2}{n-1}\left(\frac{n+1}{2c}\right)^{\frac{1}{2}}\right]^{\frac{2}{n-1}} (t_0-t)^{-\frac{2}{n-1}}, \tag{63}$$

which of course is not a *proper* solution. Thus the only positive proper solutions are

$$\theta(x) \simeq \left[\frac{(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}} x^{-\frac{\sigma+2}{n-1}}. \tag{64}$$

* The use of this theorem, however, does not extend easily to equations in which x^{σ} is replaced by $f(x)$, where $f(x) \sim x^{\sigma}$.

All other solutions are such that $\theta(x)$ vanishes or $\theta(x) \to \infty$ for finite x. A closer examination of the solutions for which $v(t) \sim 1$ shows that they are a family depending on one arbitrary constant only. It is easy to prove that *there is just one member of this family through any point and that it separates the solutions for which $\theta(x) \to \infty$ for finite x from those for which $\theta(x)$ vanishes for finite x.*

Case I (bis). $\sigma+2=0$, $2\sigma+n+3>0$, $\sigma+n+1>0$.

Writing $x = e^t$ the equation becomes at once

$$\theta''-\theta'-\theta^n=0. \tag{65}$$

It is easily verified that the nature and arrangement of the solutions remain the same except that the form (64) is replaced by

$$\theta \sim \left[\frac{1}{(n-1)\log Cx}\right]^{\frac{1}{n-1}}, \tag{66}$$

where C is the single constant of integration allowed to the family of proper solutions. The form (63) for the solutions that tend to infinity as $t \to t_0$ survives in the form

$$\theta(t) \sim \left[\frac{2}{n-1}\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\right]^{\frac{2}{n-1}}\left(\frac{1}{t_0-t}\right)^{\frac{2}{n-1}}. \tag{67}$$

Cases II, IV. $\sigma+2<0$, $2\sigma+n+3><0$, $\sigma+n+1>0$.

Here $\theta=[\varpi(1-\varpi)]^{\frac{1}{n-1}}x^{\varpi}v$, $\varpi=-\dfrac{\sigma+2}{n-1}>0$, $v>0$,

$$v''-bv'-c(v+v^n)=0 \qquad (c>0).$$

The sign of b is unimportant. Again the nature and arrangement of the solutions remains the same as in Case I, except that the family of proper solutions depending on one arbitrary constant has the asymptotic form

$$\theta(x)=A+\frac{A^n x^{\sigma+2}\alpha}{(\sigma+1)(\sigma+2)} \qquad (\alpha \sim 1). \tag{68}$$

Case III. $\sigma+2<0$, $2\sigma+n+3=0$, $\sigma+n+1>0$.

The nature and arrangement of the solutions remain the same except that (68) is replaced by the exact form

$$\theta(x)=\left[\frac{\frac{1}{2}(n+1)C^2}{\{1-C^2x^{-\frac{1}{2}(n-1)}\}^2}\right]^{\frac{1}{n-1}}, \tag{69}$$

C being a constant of integration.

Case V. $\sigma+2<0,\ 2\sigma+n+3<0,\ \sigma+n+1=0.$

Here $$\theta = xv, \qquad v>0,$$
$$v''+v'-v^n=0,$$

but the results are identical with those of Cases II and IV.

Case VI. $\sigma+2<0,\ 2\sigma+n+3<0,\ \sigma+n+1<0.$

Here $$\theta=[\varpi(\varpi-1)]^{\frac{1}{n-1}}x^{\varpi}v \qquad \varpi>1,\ v>0,$$
$$v''+bv'+c(v-v^n)=0 \qquad b,\ c>0,$$

and for the first time there is some variety of solutions possible. Study of the equation in v shows that $v\to 1$ monotonically and $v\to 0$ are both possible, but that if $v\to\infty$ the form is (63) and the solution is not a proper one. The alternative $v\to 0$ leads to

$$\theta(x)=Ax+B+\alpha\frac{x^{\sigma+n+2}}{(\sigma+n+1)(\sigma+n+2)} \qquad (\alpha\sim 1). \tag{70}$$

A, B are two constants of integration, but B is only *effectively* present if $\sigma+n+2<0$. The alternative $v\to 1$ (with $v=1$) leads of course to

$$\theta(x)\sim\left[\frac{(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}}x^{-\frac{\sigma+2}{n-1}}, \tag{71}$$

which represents a family of solutions depending on one constant of integration. The member of this family through any point separates the solutions through that point which tend to infinity for finite x from the solutions of the family (70), some of which, of course, may cross or attempt to cross the axis $\theta=0$ and so leave the region under discussion.

We may summarize these results in Table III.

Table III

The form as $x\to\infty$ of positive proper solutions of

$$\theta''-x^{\sigma}\theta^n=0 \qquad (n=p/q>1).$$

All the solutions are ultimately monotonic. They cease to be positive proper solutions either by crossing or attempting to cross the axis $\theta=0$ or by becoming infinite for finite x, in which case they take the asymptotic form

$$\theta(x)\sim x^{-\frac{\sigma+2}{n-1}}\left[\frac{2}{n-1}\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\right]^{\frac{2}{n-1}}\left[\log\frac{x_0}{x}\right]^{-\frac{2}{n-1}}. \tag{Y}$$

Positive proper solutions have the following forms:

Case I. $\sigma+2>0,\ 2\sigma+n+3>0,\ \sigma+n+1>0.$

$$\theta(x) \cong \left[\frac{(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}} x^{-\frac{\sigma+2}{n-1}}, \qquad (Z)$$

with one arbitrary constant; curves separate solutions for which $\theta(x)\to\infty$ *from those for which* $\theta(x)\to 0$ *as* $x\to x_0$.

Case I (bis). $\sigma+2=0$. *As for I, except that*

$$\theta(x) \sim \left[\frac{1}{(n-1)\log(Cx)}\right]^{\frac{1}{n-1}}.$$

Cases II–V. $\sigma+2<0,\ \sigma+n+1\geqslant 0$. *As for I, except that*

$$\theta(x) = A + \frac{A^n x^{\sigma+2}}{(\sigma+1)(\sigma+2)}\alpha \qquad (\alpha\sim 1).$$

When $2\sigma+n+3=0$,

$$\theta(x) = \left[\frac{\frac{1}{2}(n+1)C^2}{\{1-C^2x^{-\frac{1}{2}(n-1)}\}^2}\right]^{\frac{1}{n-1}}.$$

Case VI. $\sigma+n+1<0$.

$$\theta(x) \cong \left[\frac{(\sigma+2)(\sigma+n+1)}{(n-1)^2}\right]^{\frac{1}{n-1}} x^{-\frac{\sigma+2}{n-1}}, \qquad (Z)$$

with one arbitrary constant, or

$$\theta(x) = Ax+B+\frac{A^n x^{\sigma+n+2}}{(\sigma+n+1)(\sigma+n+2)}\alpha \qquad (\alpha\sim 1), \qquad (\Omega)$$

with two constants of integration A, B. *Curves of the family* (Z) *separate solutions for which* $\theta(x)\to\infty$ *as* $x\to x_0$ *from members of the family* (Ω).

A NOTE ON THE HYPERGEOMETRIC AND BESSEL'S EQUATIONS (II)

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1. Introduction

In part I of this Note* we called attention to certain elementary propositions in the theory of the generalized hypergeometric equation

$$\prod(\delta-a)y = x^m \prod(\delta+b)y, \tag{1}$$

which enable us to transform equations of this type, by substitutions of the form

$$y = \prod(\delta-c)z, \tag{2}$$

into equations of similar or simpler character in z.

Our arguments were based on the main theorem I (2), namely:

If z is a solution of the equation

$$(\delta-a)f(\delta)z = x^m g(\delta)z,$$

then

$$y = \prod_{s=0}^{r-1}(\delta-a-sm)z$$

is a solution of the equation

$$(\delta-a-rm)f(\delta)y = x^m g(\delta)y.$$

In part I we applied this theorem to a special class of equations (1), namely, those of the generalized Bessel's type

$$\prod(\delta-a)y = x^m y, \tag{3}$$

employing it mainly to throw light on the classification of such equations by the types of function entering into their general solution.

In this part II we undertake a similar inquiry for the more general equations (1), in which, however, some of the simplicity of the earlier inquiry will be lacking for the following reason. The substitution (2) derives the complete solution of the y-equation from the complete solution of the z-equation, unless either the order of the z-equation is less than that of the y-equation, or else a solution of the z-equation is annihilated by the operator $\prod(\delta-a-sm)$.

With equations of the generalized Bessel's type (3) neither of these possibilities occur. On the one hand the order of the y-equation is

* *Quart. J. of Math.* (Oxford), 1 (1930), 186–95. References to results therein will be given in the form: I (4).

the difference of the orders of the δ-operators on the two sides of the equation, which is invariant for substitutions of the type (2). The z-equation cannot therefore be of lower order than the y-equation, and in cases of interest is of the same order, when consequently it is also of the generalized Bessel's form (3). Therefore, on the other hand, no solution of the z-equation can be annihilated by the operator $\prod(\delta-a)$, for this operator annihilates only *finite* sums of the form $\sum A_c x^c$, whereas the generalized Bessel's equations have no finite solutions of this form.

Thus, as we have found in part I, the complete solution of the z-equation leads at once to the complete solution of the y-equation. With equations of the more general type (1) this need not always be the case, and we must then fill in the 'missing' solution (or solutions) from other sources.

2. Amplification of Theorem I (2)

We begin by extending the theorem I (2) into a form which takes account of these missing solutions.

We evidently lack a solution y, if and only if some z is a finite sum of powers

$$x^{a+sm} \qquad (s = 0, 1, \ldots, r-1).$$

For this the necessary and sufficient condition is that $g(\delta)$ have a factor $\delta-a-sm$, where s has one of the above values. If there are several such factors, suppose $s = q$ is that with greatest s, and consider the equation

$$(\delta-a-rm)f(\delta)y = x^m(\delta-a-qm)g_1(\delta)y. \tag{4}$$

The substitution

$$y = \prod_{s=q}^{r-1}(\delta-a-sm)z$$

now gives for z the equation

$$(\delta-a-qm)f(\delta)z = x^m(\delta-a-qm)g_1(\delta)z$$

or sufficiently, if $z_1 \equiv (\delta-a-qm)z$,

$$f(\delta)z_1 = x^m g_1(\delta)z_1.$$

In this case, then, as compensation for missing a solution, we have been able to lower the order of the equation.

To replace the missing solution we observe that the full equation for z_1 is actually

$$\prod_{s=q+1}^{r}(\delta-a-sm)\{f(\delta)z_1 - x^m g_1(\delta)z_1\} = 0,$$

or sufficiently,

$$[\delta-a-m(q+1)]\{f(\delta)z_1-x^m g_1(\delta)z_1\}=0,$$

which has the first integral

$$f(\delta)z_1=x^m g_1(\delta)z_1+Cx^{a+m(q+1)}, \tag{5}$$

where C is some constant. The substitution connecting y, z is now

$$y=\prod_{s=q+1}^{r-1}(\delta-a-sm)z_1, \tag{6}$$

and z_1 is annihilated by the operator on the right, only if it be a sum of powers

$$x^{a+m(q+1)},\ldots,x^{a+m(r-1)}.$$

If such a z_1 be a solution of (5), then, whether C be zero or not, the last of the above powers must be annihilated by $g_1(\delta)$. Thus $g(\delta)$ must have had a factor $\delta-a-sm$ with $s\geqslant q+1$, which is contrary to our hypothesis about q. Consequently the complete solution of (5) with $C\neq 0$ gives, through (6), the complete solution of (4). Replacing g_1, z_1 by g, z we may enunciate this result as follows:

(1) *If q, r, s be integers such that $q<r$ and $g(\delta)$ have no factor $\delta-a-sm$, where $q<s<r$, then the complete solution of the equation*

$$(\delta-a-rm)f(\delta)y=x^m(\delta-a-qm)g(\delta)y$$

is given by

$$y=\prod_{s=q+1}^{r-1}(\delta-a-sm)z,^*$$

where z is the complete solution of the equation

$$f(\delta)z=x^m g(\delta)z+Cx^{a+m(q+1)}$$

and C is a constant other than zero.

If $f(\delta)$, $g(\delta)$ themselves have each a factor of the type $\delta-a'-r'm$, $\delta-a'-q'm$, where q', r' are subject to the conditions stated in (1) for q, r, then we can once again apply the procedure of (1) to diminish the order of the equation, adding a further term independent of z on the right. The term $Cx^{a+q(m+1)}$ already on the right is replaced by some X defined by an equation

$$\prod_{s=q'+1}^{r'}(\delta-a'-sm)X=Cx^{a+m(q+1)}.$$

In general it is enough to write

$$X=C'x^{a+m(q+1)}.$$

* With the usual convention, of course, that a product of no factors is unity.

But, if the operator on the left contain a factor $\delta-a-m(q+1)$, we have to take X of the form

$$X = Cx^{a+m(q+1)}\log x.$$

We can clearly proceed in this way so long as $f(\delta)$, $g(\delta)$ possess corresponding factors of the requisite forms.*

We can replace (**1**) by the more general theorem:

(**2**) *If q, r, s be integers such that $q<r$ and $g(\delta)$ have no factor $\delta-a-sm$, where $q<s<r$, and if p be any positive integer, then the complete solution of the equation*

$$\prod_{s=0}^{p=1}(\delta-a-sm-rm)f(\delta)y = x^{mp}(\delta-a-qm)g(\delta)y$$

is given by

$$y = \prod_{t=q+1}^{r-1}(\delta-a-tm)z,$$

where z is the complete solution of the equation

$$\prod_{s=1}^{p-1}(\delta-a-sm-qm)f(\delta)z = x^{mp}g(\delta)z + Cx^{a+m(q+p)}$$

and C is a constant other than zero.

We suppress the proof, since no new principle arises.

There is a theorem, complementary to (**1**):

(**3**) *If r is a positive integer and z any solution of the equation*

$$(\delta-a)f(\delta)z = x^m(\delta-a-rm)g(\delta)z,$$

then

$$y = \prod_{s=0}^{r}(\delta-a-sm)z$$

is a solution of the equation

$$f(\delta)y = x^m g(\delta)y.$$

In this case the order of the z-equation exceeds by unity that of the y-equation, and one solution z must be annihilated by the operator $\prod(\delta-a-sm)$. We see from the form of the z-equation that it has a finite solution led by x^a and terminated by x^{a+rm}. This is evidently the solution annihilated.

In the same way that (**2**) is the generalization of (**1**), so we have the generalization of (**3**):

* If there are several such factors of $f(\delta)$ mutually congruent to modulus m, it is necessary to proceed in order of descending r to avoid confusion and possible vitiation of the argument.

(4) *If r, p are positive integers and z is any solution of the equation*

$$\prod_{s=0}^{p-1}(\delta-a-sm)f(\delta)z = x^{mp}(\delta-a-rm)g(\delta)z,$$

then

$$y = \prod_{s=0}^{r}(\delta-a-sm)z$$

is a solution of the equation

$$\prod_{s=1}^{p-1}(\delta-a-rm-sm)f(\delta)y = x^{mp}g(\delta)y.$$

3. Equations soluble by elementary functions

In analogy with I (11) we now consider the equation

$$(\delta-a_1)(\delta-a_2)\ldots(\delta-a_m)y = x^m(\delta+b_1)(\delta+b_2)\ldots(\delta+b_m)y,$$

where the order of the δ-operators on each side is equal to the index of x^m and the differences a_r-a_s, b_r-b_s are all integers incongruent to modulus m. We may write this equation more conveniently as

$$\prod_{r=0}^{m-1}(\delta-a-r-mN_r)y = x^m\prod_{r=0}^{m-1}(\delta+b+r+mN'_r)y, \tag{7}$$

where N_r, N'_r are integers of either sign or zero, and a, b are any real numbers. By I (4), this is satisfied by some

$$y = \prod(\delta-c)z,$$

where z is a solution of

$$\prod_{r=0}^{m-1}(\delta-a-r-mN)z = x^m\prod_{r=0}^{m-1}(\delta+b+r+mN')z, \tag{8}$$

and N, N' are the algebraically least integers of the sets (N_r), (N'_r). By I (6) this equation is satisfied by the solution of

$$(\delta-a-mN)z = x(\delta+b+mN')z,$$

and therefore (8) has the m independent solutions

$$z = x^{a+mN}(1-\omega x)^{-a-b-m(N+N')} \qquad (\omega^m = 1). \tag{9}$$

Unless $a+b+m(N+N')$ be a negative integer or zero, the expressions (9) for z are not expressible as finite sums of powers of x and therefore are not annihilated by the operator $\prod(\delta-c)$. We can thus assert that

(5) *In general, the equation* (7) *is soluble by algebraic functions.*

In the exceptional case, when $a+b+m(N+N')$ is a negative integer or zero, the substitution $y = x^a y'$ in (7) and a cyclic rearrangement of factors on the right make it enough to consider, in place of (7), the equation

$$\prod_{r=0}^{m-1}(\delta-r-mN_r)y = x^m\prod_{r=0}^{m-1}(\delta+r+mN'_r)y. \tag{10}$$

The exceptional case is now that in which $N+N' \leqslant 0$. If we write the right-hand side of (10) in the form

$$x^m \prod_{r=0}^{m-1} (\delta - r + m + mN'_{m-r})y$$

and apply (1), we see that solutions are 'missing' only when

$$N_r > -1 - N'_{m-r}.$$

Let us then divide the integers $r = 0, 1, \ldots, m-1$ into two groups (s), (t), according as

$$N_s + N'_{m-s} \geqslant 0, \qquad N_t + N'_{m-t} < 0.$$

By (1) and the subsequent discussion it appears that the complete solution of (10) is given by

$$y = \prod_s \prod_{k=-N'_{m-s}}^{N_s - 1} (\delta - s - km)z,$$

where z is the complete solution of

$$\prod_t (\delta - t - mN_t)z = x^m \prod_t (\delta - t + m + mN'_{m-t})z + \sum_s C_s x^{-s-mN'_{m-s}}. \quad (11)$$

No logarithms appear, since the factors $\delta - s - mN_s$ that have been removed from the right are mutually incongruent to modulus m.

Operation on (11) with the operator

$$\prod_t \prod_{k=N_t+1}^{-N'_{m-t}-1} (\delta - t - km)$$

gives the equation

$$(1-x^m) \prod_t \prod_{k=N_t}^{-N'_{m-t}-1} (\delta - t - km)z = \sum_s C'_s x^{-s-mN'_{m-s}},$$

that is to say,

$$\prod_{k,t} (\delta - t - km)z = \sum_s \frac{C'_s x^{-s-mN'_{m-s}}}{1-x^m}. \quad (12)$$

No terms have been lost on the right, since $s \not\equiv t \pmod m$. The complete solution of (12) thus includes the complete solution of (11). In (12) the 'complementary function, is a sum of powers of x; a 'particular integral' evidently introduces, besides powers of x, the logarithms $\log(1-\omega x)$, where $\omega^m = 1$. Terms involving $\log x$ could also conceivably appear in the particular integral. In this case z must include a term

$$x^{t+km} \log x,$$

which gives rise to a term x^{t+km} on the right of (12). But there are no such terms, since t, $m-s$ are never congruent to modulus m. Thus no logarithms appear except those of $(1-\omega x)$, and the complete

solution of (12) involves at most these and a finite set of powers of x. The same is true of (7) in the exceptional case, and we may thus sum up and say

(6) *The differential equation*

$$\prod_{r=0}^{m-1}(\delta-a-r-mN_r)y = x^m \prod_{r=0}^{m-1}(\delta+b+r+mN'_r),$$

where N_r, N'_r are integers of either sign or zero, and a, b are any real numbers, is, in general, soluble in algebraic functions, and in any case is soluble in algebraic functions together with the logarithms $\log(1-\omega x)$, *where* $\omega^m = 1$.

4. Legendre's equation

In illustration we may apply the preceding theory to a discussion of Legendre's equation

$$\delta(\delta-1)y = x^2(\delta-n)(\delta+n+1)y \tag{13}$$

in the case in which n is a positive integer. If in (2) we write

$$a = 0 = r, \qquad m = 1, \qquad p = 2, \qquad q = -(n+1),$$
$$f(\delta) \equiv 1, \qquad g(\delta) \equiv (\delta-n),$$

we obtain the solutions

$$D^n(x^2-1)^n, \qquad D^n\left\{(x^2-1)^n \int_x^\infty \frac{dt}{(t^2-1)^{n+1}}\right\},$$

recognizable* as constant multiples of $P_n(x)$, $Q_n(x)$.

Since n, $-(n+1)$ are of opposite parity, Theorem (6) is applicable to Legendre's equation, which, as is otherwise known, is soluble in terms of algebraic functions together with the logarithms $\log(1\pm x)$. Various formulae† expressing $Q_n(x)$ more or less explicitly in algebraic and logarithmic parts may be obtained by transforming Legendre's equation into the symbolic form

$$y = x^{-n}\frac{\delta(\delta-1)\ldots(\delta-n+1)}{\delta(\delta-2)\ldots(\delta-2n)}\,\frac{C_1 x}{1-x^2}, \tag{14}$$

and suitably manipulating the fractional operator.

If we apply (4) to (13), we obtain the known formula

$$D^{n+1}Q_n(x) = 2^n n!(1-x^2)^{-(n+1)},$$

* For $Q_n(x)$ compare Whittaker and Watson, *Modern Analysis* (1920), 15.31, Ex. 1. The proof there given is essentially the method of the present note.

† Cf. Whittaker and Watson, loc. cit., p. 333, Ex. 29.

determining the numerical factor from a comparison of leading coefficients.

We may obtain certain less usual formulae for P_n, Q_n, if we separate the cases of n even, n odd. Write first $n = 2p$ and in (1) set

$$a = -(2p+1), \qquad q = 0, \qquad r = p+1,$$
$$f(\delta) \equiv \delta, \qquad g(\delta) \equiv \delta - 2p.$$

We then have the complete solution of Legendre's equation,

$$y = \prod_{s=1}^{p} (\delta + 2p + 1 - 2s)z = x(x^{-1}D)^p x^{2p-1} z = x^{-2p-1}(x^3 D)^p xz,$$

where z is the complete solution of the equation

$$\delta z = x^2(\delta - 2p)z + Cx^{-2p+1}, \tag{15}$$

and so

$$z = C'(x^2-1)^{-p} + C(x^2-1)^p \int_x^\infty x^{-2p}(x^2-1)^{-p-1}\, dx. \tag{16}$$

From the coefficient of C' we obtain, on comparison of leading coefficients, the formulae

$$2^p p! P_{2p}(x) = x(x^{-1}D)^p x^{2p-1}(x^2-1)^p = x^{-2p-1}(x^3 D)^p x(x^2-1)^p,$$

which may also be written as

$$p! P_{2p}(x^{\frac{1}{2}}) = x^{\frac{1}{2}} D^p \{x^{p-\frac{1}{2}}(x-1)^p\},$$
$$p! P_{2p}(x^{-\frac{1}{2}}) = x^{p+\frac{1}{2}} D^p \{x^{-p-\frac{1}{2}}(x-1)^p\}.$$

We may express these in the form of contour-integrals as

$$P_{2p}(x^{\frac{1}{2}}) = \frac{x^{\frac{1}{2}}}{2\pi i} \int_\Gamma \frac{z^{p-\frac{1}{2}}(z-1)^p\, dz}{(z-x)^{p+1}},$$

$$P_{2p}(x^{-\frac{1}{2}}) = \frac{x^{p+\frac{1}{2}}}{2\pi i} \int_\Gamma \frac{z^{-p-\frac{1}{2}}(z-1)^p\, dz}{(z-x)^{p+1}},$$

where in both integrals Γ is a contour in the z-plane which encloses $z = x$ but not $z = 0$. These contour-integrals are evidently special cases of those which represent the hypergeometric function.

Again from the coefficient of C in (16) we similarly obtain

$$Q_{2p}(x^{\frac{1}{2}}) = \frac{(-)^p 2^{2p-1}\{(2p)!\}^2}{p!(4p)!} x^{\frac{1}{2}} D^p x^{p-\frac{1}{2}}(x-1)^p \int_x^\infty t^{-p-\frac{1}{2}}(t-1)^{-p-1}\, dt,$$

$$Q_{2p}(x^{-\frac{1}{2}}) = \frac{2^{2p-1}\{(2p)!\}^2}{p!(4p)!} x^{\frac{1}{2}+p} D^p x^{-\frac{1}{2}-p}(1-x)^p \int_0^x t^{2p-\frac{1}{2}}(1-t)^{-p-1}\, dt.$$

If n still equals $2p$ and we apply (3) to (13), we see that, if y is any solution of Legendre's equation of order $2p$, then

$$z = \prod_{s=0}^{p} (\delta - 2s) y \tag{17}$$

is a solution of

$$(\delta - 1)z = x^2(\delta + 2p + 1)z. \tag{18}$$

Since (18) is satisfied by

$$z = x(x^2-1)^{-p-1}$$

and P_{2p} is annihilated by the operator on the right of (17), we have, on comparison of leading coefficients,

$$\prod_{s=0}^{p} (\delta - 2s) Q_{2p}(x) = (-)^{p+1} 2^p p! x (x^2-1)^{-p-1},$$

which may be written in the equivalent forms

$$D^{p+1} Q_{2p}(x^{\frac{1}{2}}) = (-)^{p+1} \tfrac{1}{2} p! x^{-p-\frac{1}{2}} (x-1)^{-p-1},$$
$$D^{p+1} x^p Q_{2p}(x^{-\frac{1}{2}}) = (-)^{p+1} \tfrac{1}{2} p! x^{p-\frac{1}{2}} (x-1)^{-p-1}.$$

Corresponding results for $n = 2p+1$ are

$$p! P_{2p+1}(x^{\frac{1}{2}}) = D^p \{x^{p+\frac{1}{2}} (x-1)^p\},$$
$$p! P_{2p+1}(x^{-\frac{1}{2}}) = x^{p+1} D^p \{x^{-p-\frac{1}{2}} (x-1)^p\},$$

$$Q_{2p+1}(x^{\frac{1}{2}}) = \frac{(-)^p 2^{2p} \{(2p+1)!\}^2}{p!(4p+2)!} D^p x^{p+\frac{1}{2}} (x-1)^p \int_x^\infty x^{-p-\frac{1}{2}} (x-1)^{-p-1} \, dx,$$

$$Q_{2p+1}(x^{-\frac{1}{2}}) = \frac{2^{2p} \{(2p+1)!\}^2}{p!(4p+2)!} x^{p+1} D^p x^{-p-\frac{1}{2}} (1-x)^p \int_0^x x^{2p+\frac{1}{2}} (1-x)^{-p-1} \, dx,$$

$$D^{p+1} x^{-\frac{1}{2}} Q_{2p+1}(x^{\frac{1}{2}}) = \tfrac{1}{2} p! x^{-p-\frac{1}{2}} (1-x)^{-p-1},$$
$$D^{p+1} x^{p+\frac{1}{2}} Q_{2p+1}(x^{-\frac{1}{2}}) = \tfrac{1}{2} p! x^{p+\frac{1}{2}} (1-x)^{-p-1}.$$

In similar fashion we can obtain formulae for the associated Legendre functions by considering the differential equation

$$\delta(\delta - 1)y = x^2(\delta + m - n)(\delta + m + n + 1)y$$

satisfied by

$$(1-x^2)^{-\frac{1}{2}m} P_n^m(x).$$

SOME SELF-RECIPROCAL FUNCTIONS

By G. N. WATSON (*Birmingham*)

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1. THIS note contains a few minor developments of a recent paper* entitled 'Self-reciprocal functions' by Hardy and Titchmarsh. The problems which I discuss have arisen out of a question which was put to me by Mr. E. G. Phillips; he had encountered the formula†

$$1+2\sum_{n=1}^{\infty}J_0(nx)=4\sum_{n=0}^{p}{}'\frac{1}{\sqrt{(x^2-4n^2\pi^2)}}, \tag{1}$$

where p is the greatest integer such that $2\pi p\leqslant x$ and the dash indicates that the first term of the sum is to be halved; and his question was whether an analogous formula existed for the series obtained from (1) by changing $J_0(nx)$ into $K_0(nx)$.

In this paper I first consider the more general series

$$\sum_{n=1}^{\infty}(\tfrac{1}{2}nz)^{\nu}K_{\nu}(nz) \tag{2}$$

and I transform it into an infinite series which is a natural modification of the sum on the right in (1); it is then a trivial matter to obtain the transformation in the special case $\nu=0$ by a limiting process. It is, however, more interesting to observe that a slight modification of the series (2) yields a function $F(z)$ which is self-reciprocal in a certain Hankel transform, i.e. such that

$$F(z)=\int_0^{\infty}\sqrt{(yz)}J_{\mu}(yz)F(y)\,dy.$$

This result suggested to me a further problem, namely, whether, if a series

$$f(x)+f(2x)+f(3x)+\ldots$$

is self-reciprocal in a Hankel transform, it is possible to construct a kernel $\varpi(x)$ for which the individual function $f(x)$ is self-reciprocal, so that

$$f(x)=\int_0^{\infty}\varpi(xy)f(y)\,dy$$

* *Quart. J. of Math.* (Oxford), 1 (1930), 196–231.

† This formula is given by Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (1904), 336, with its extension to functions of order ν.

I give a formal solution of this problem and of the corresponding problem for the series

$$f(x)-f(3x)+f(5x)-\ldots;$$

and lastly I investigate a transform which I believe to be new.

2. We proceed to the discussion of the series (2) in which it is supposed that z and ν are complex with $R(z)>0$, $R(\nu)>0$.

We take Poisson's summation formula* in a form applicable to an even function $f(t)$, namely,

$$f(0)+2\sum_{n=1}^{\infty}f(n)=2\int_0^{\infty}f(t)\,dt+4\sum_{n=1}^{\infty}\int_0^{\infty}f(t)\cos 2\pi nt\,dt, \qquad (3)$$

and, for the specified domains of ν and z, we may define $f(t)$ for positive values of t by the formula

$$f(t)=(\tfrac{1}{2}tz)^{\nu}K_{\nu}(tz).$$

On the left of (3) we have to interpret $f(0)$ as $\lim_{t\to+0}f(t)=\frac{1}{2}\Gamma(\nu)$.

To evaluate the integrals on the right, we observe that

$$\cos at=\sqrt{(\tfrac{1}{2}\pi at)}\,J_{-\frac{1}{2}}(at),$$

and that, by a known formula,†

$$\int_0^{\infty}\frac{J_{\mu}(at)K_{\nu}(tz)}{t^{\lambda}}dt=\frac{a^{\mu}\Gamma(\frac{1}{2}\mu-\frac{1}{2}\lambda+\frac{1}{2}\nu+\frac{1}{2})\Gamma(\frac{1}{2}\mu-\frac{1}{2}\lambda-\frac{1}{2}\nu+\frac{1}{2})}{2^{\lambda+1}z^{\mu-\lambda+1}\Gamma(\mu+1)}\times$$
$$\times F\left(\frac{\mu-\lambda+\nu+1}{2},\frac{\mu-\lambda-\nu+1}{2};\mu+1;-\frac{a^2}{z^2}\right).$$

Hence, by taking $\lambda=-\nu-\frac{1}{2}$ and $\mu=-\frac{1}{2}$, we get

$$\int_0^{\infty}(\tfrac{1}{2}tz)^{\nu}K_{\nu}(tz)\cos at\,dt=\frac{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})z^{2\nu}}{2(z^2+a^2)^{\nu+\frac{1}{2}}}.$$

We now deduce immediately from (3) that

$$\tfrac{1}{2}\Gamma(\nu)+2\sum_{n=1}^{\infty}(\tfrac{1}{2}nz)^{\nu}K_{\nu}(nz)$$
$$=\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})z^{2\nu}\left[\frac{1}{z^{2\nu+1}}+2\sum_{n=1}^{\infty}\frac{1}{(z^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\right], \qquad (4)$$

provided that $R(z)>0$, $R(\nu)>0$.

[I remark that I owe this proof to the kindness of a referee. My

* For a proof of the formula under conditions which are fulfilled in the example which we are considering, see J. R. Wilton, *Journal London Math. Soc.* 5 (1930), 276–9.

† See my *Theory of Bessel Functions* (1922), p. 410, formula (1).

original proof consisted in taking the formula

$$(\tfrac{1}{2}nz)^{\nu}K_{\nu}(nz)=\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\nu)}\int_1^{\infty}\frac{e^{-nzt}}{(t^2-1)^{\nu+\frac{1}{2}}}\,dt,$$

transforming the integral on the right into a loop-integral valid if $R(\nu)>0$, and so expressing the left-hand side of (4) as a contour integral which is evaluated by considering poles on the imaginary axis; my own proof was longer than the proof just given.]

3. We now obtain a modification of (4) valid when the restriction on ν is removed. Let M be any positive integer and consider

$$\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})z^{2\nu}\times$$
$$\times\left[\frac{1}{z^{2\nu+1}}+2\sum_{n=1}^{\infty}\left\{\frac{1}{(z^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}-\sum_{m=0}^{M-1}\binom{-\nu-\frac{1}{2}}{m}\frac{z^{2m}}{(2n\pi)^{2\nu+2m+1}}\right\}+\right.$$
$$\left.+2\sum_{m=0}^{M-1}\binom{-\nu-\frac{1}{2}}{m}\frac{z^{2m}\zeta(2\nu+2m+1)}{(2\pi)^{2\nu+2m+1}}\right].$$

Since the general term of the infinite series is $O(n^{-2\nu-2M-1})$ when n is large, the expression, *qua* function of ν, is analytic throughout the domain $R(\nu)>-M$ with the possible exception of the poles of $\Gamma(\nu+\frac{1}{2})$ and of the zeta functions; and, in the more restricted domain in which $R(\nu)>0$, the expression reduces to the expression on the right of (4).

Consequently, by the theory of analytic continuation, we have

$$\tfrac{1}{2}\Gamma(\nu)+2\sum_{n=1}^{\infty}(\tfrac{1}{2}nz)^{\nu}K_{\nu}(z)=\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})z^{2\nu}\times$$
$$\times\left[\frac{1}{z^{2\nu+1}}+2\sum_{n=1}^{\infty}\left\{\frac{1}{(z^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}-\sum_{m=0}^{M-1}\binom{-\nu-\frac{1}{2}}{m}\frac{z^{2m}}{(2n\pi)^{2\nu+2m+1}}\right\}+\right.$$
$$\left.+2\sum_{m=0}^{M-1}\binom{-\nu-\frac{1}{2}}{m}\frac{z^{2m}\zeta(2\nu+2m+1)}{(2\pi)^{2\nu+2m+1}}\right],\qquad(5)$$

not only when $R(\nu)>0$ but also when $R(\nu)>-M$, except when ν has any of the values $0, -\frac{1}{2}, -1, -\frac{3}{2},\ldots, -M+\frac{1}{2}$; and it is, of course, supposed that $R(z)>0$.

We now consider what happens when $\nu=0$. In (5) take $M=1$ and proceed to the limit. We get

$$2\sum_{n=1}^{\infty}K_0(nz)=\pi\left[\frac{1}{z}+2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{(z^2+4n^2\pi^2)}}-\frac{1}{2n\pi}\right\}\right]+$$
$$+\lim_{\nu\to 0}\left[\frac{2\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{(2\pi)^{2\nu+1}}z^{2\nu}\zeta(2\nu+1)-\frac{\Gamma(\nu)}{2}\right],$$

and hence

$$2\sum_{n=1}^{\infty} K_0(nz) = \pi\left[\frac{1}{z}+2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{(z^2+4n^2\pi^2)}}-\frac{1}{2n\pi}\right\}\right]+ \\ +\gamma+\log(\tfrac{1}{2}z)-\log(2\pi), \qquad (6)$$

where γ is Euler's constant and $R(z)>0$.

It seems unnecessary to work out any of the other limiting cases in which $\nu \to -m$ $(m = 1, 2, 3, \ldots)$.

4. A comparison of formula (4) with the result*

$$\int_0^{\infty} \frac{y^{\mu+1}J_\mu(xy)}{(y^2+k^2)^{\nu+\frac{1}{2}}}\,dy = \frac{(\frac{1}{2}x)^{\nu-\frac{1}{2}}k^{\mu-\nu+\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}K_{\mu-\nu+\frac{1}{2}}(kx) \qquad (7)$$

(and the reciprocal of this result in the Hankel transform) inevitably suggests the construction of the function reciprocal to the function defined by the series on either side of (4). Accordingly, let

$$F_\nu(z) = \tfrac{1}{2}\Gamma(\nu)+2\sum_{n=1}^{\infty}(\tfrac{1}{2}nz)^\nu K_\nu(nz) \\ = \Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})z^{2\nu}\left[\frac{1}{z^{2\nu+1}}+2\sum_{n=1}^{\infty}\frac{1}{(z^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\right], \qquad (8)$$

where $R(z)>0$ and $R(\nu)>0$; and consider

$$\int_0^{\infty}\sqrt{(xy)}J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}F_\nu(ay)\,dy \qquad (x>0,\ a>0).$$

We proceed to examine the permissibility of substituting the second series in (8) for $F_\nu(ay)$ and then integrating term by term. We first take

$$R(\nu)>0, \qquad -1<R(\mu)<R(2\nu-2). \qquad (9)$$

Then $$\left|\frac{y^{\mu+1}J_\mu(xy)}{(a^2y^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\right| \leqslant \frac{|y^{\mu+1}J_\mu(xy)|}{(2n\pi)^{R(2\nu+1)}}, \qquad (0\leqslant y\leqslant 1)$$

and $$\left|\frac{y^{\mu+1}J_\mu(xy)}{(a^2y^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\right| \leqslant \frac{|y^{-1}J_\mu(xy)|}{a^{R(\mu+2)}(2n\pi)^{R(2\nu-\mu-1)}} \qquad (y\geqslant 1).$$

Since $\int_0^1 |y^{\mu+1}J_\mu(xy)|\,dy$ and $\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^{R(2\nu+1)}}$ are convergent, we have

* This result and its reciprocal are given in my *Theory of Bessel Functions* (1922), 434 and 410.

$$\int_0^1 \sqrt{(xy)}J_\mu(xy)(ay)^{\mu+\frac{1}{2}}\left\{\sum_{n=1}^{\infty}\frac{1}{(a^2y^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\right\}dy$$

$$=\sum_{n=1}^{\infty}\int_0^1 \sqrt{(xy)}J_\mu(xy)\frac{(ay)^{\mu+\frac{1}{2}}}{(a^2y^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\,dy,$$

from the first of the inequalities; also, since $\int_1^\infty |y^{-1}J_\mu(xy)\,dy|$ and $\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^{R(2\nu-\mu-1)}}$ are convergent, it follows from the second of the inequalities that the order of the summation and of the integration from 1 to ∞ may be interchanged.

Combining these results, we get

$$\int_0^\infty \sqrt{(xy)}J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}\left\{F_\nu(ay)-\frac{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{ay}\right\}dy$$

$$=\sum_{n=1}^{\infty}\int_0^\infty \sqrt{(xy)}J_\mu(xy)\frac{2\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})(ay)^{\mu+\frac{1}{2}}}{(a^2y^2+4n^2\pi^2)^{\nu+\frac{1}{2}}}\,dy,$$

so that, by (7),

$$\int_0^\infty \sqrt{(xy)}J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}\left\{F_\nu(ay)-\frac{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{ay}\right\}dy$$

$$=2\frac{\sqrt{(2\pi)}}{a}\left(\frac{x}{2a}\right)^{2\nu-\mu-\frac{1}{2}}\sum_{n=1}^{\infty}\left(\frac{n\pi x}{a}\right)^{\mu-\nu+\frac{1}{2}}K_{\mu-\nu+\frac{1}{2}}\left(\frac{2n\pi x}{a}\right).$$

Now the series on the right is an analytic function of both μ and ν in any finite domains of values of μ and ν; and by dividing the range of integration on the left into the ranges $(0,1)$ and $(1,\infty)$ and using the second and first expressions in (8) in the respective ranges, we see that each of the integrals is an analytic function of both μ and ν in the domains specified by the inequalities

$$R(2\nu)>R(\mu+\tfrac{1}{2})>R(\nu)>0. \tag{10}$$

Also

$$\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})\int_0^\infty \sqrt{(xy)}J_\mu(xy)(ay)^{\mu-2\nu-\frac{1}{2}}\,dy=\frac{\sqrt{(2\pi)}}{2a}\left(\frac{x}{2a}\right)^{2\nu-\mu-\frac{1}{2}}\Gamma(\mu-\nu+\tfrac{1}{2}),$$

and so

$$\int_0^\infty \sqrt{(xy)}J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}F_\nu(ay)\,dy=\frac{\sqrt{(2\pi)}}{a}\left(\frac{x}{2a}\right)^{2\nu-\mu+\frac{1}{2}}F_{\mu-\nu+\frac{1}{2}}\left(\frac{2\pi x}{a}\right), \tag{11}$$

throughout the domains specified by (10).

The only point of the inequality $R(2\nu) > R(\mu+\frac{1}{2})$ in (10) is now to secure the convergence of the integral at the upper limit in consequence of the term $\frac{1}{2}\Gamma(\nu)$ in (8). Since

$$\tfrac{1}{2}\Gamma(\nu)\int_0^\infty \sqrt{(xy)}J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}\,dy$$

exists as a 'generalized integral' written

$$G\int_0^\infty = \lim_{\lambda\to+0}\int_0^\infty e^{-\lambda y},$$

we find that, if $R(\mu+\frac{1}{2}) > R(\nu) > 0,$ (10a)

then

$$G\int_0^\infty \sqrt{(xy)}J_\mu(xy)(ay)^{\mu-2\nu+\frac{1}{2}}F_\nu(ay)\,dy = \frac{\sqrt{(2\pi)}}{a}\left(\frac{x}{2a}\right)^{2\nu-\mu-\frac{1}{2}}F_{\mu-\nu+\frac{1}{2}}\left(\frac{2\pi x}{a}\right). \tag{11a}$$

The results (11) and (11a) show that the functions

$$(ax)^{\mu-2\nu+\frac{1}{2}}F_\nu(ax), \qquad \frac{\sqrt{(2\pi)}}{a}\left(\frac{x}{2a}\right)^{2\nu-\mu-\frac{1}{2}}F_{\mu-\nu+\frac{1}{2}}\left(\frac{2\pi x}{a}\right)$$

are reciprocal functions in the (possibly generalized) Hankel transform of order μ.

In particular, if we take $\mu = 2\nu-\frac{1}{2}$ and $a = \sqrt{(2\pi)}$, we see that when $R(\nu) > 0$ the function $F_\nu\big(x\sqrt{(2\pi)}\big)$, that is to say

$$\tfrac{1}{2}\Gamma(\nu)+2\sum_{n=1}^\infty \{nx\sqrt{(\tfrac{1}{2}\pi)}\}^\nu K_\nu\big(nx\sqrt{(2\pi)}\big),$$

is self-reciprocal in the generalized Hankel transform of order $2\nu-\frac{1}{2}$.

5. We now give a purely formal discussion of the general problem concerning the integral equation satisfied by $f(x)$ when the series

$$f(x)+f(2x)+f(3x)+\ldots$$

is a self-reciprocal function in the Hankel transform of order ν. We write

$$\phi(s) = \int_0^\infty x^{s-1}f(x)\,dx, \qquad f(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\phi(s)x^{-s}\,ds,$$

and then

$$\int_0^\infty x^{s-1}\sum_{n=1}^\infty f(nx)\,dx = \sum_{n=1}^\infty \int_0^\infty x^{s-1}f(nx)\,dx = \sum_{n=1}^\infty n^{-s}\int_0^\infty x^{s-1}f(x)\,dx = \zeta(s)\phi(s).$$

The Hardy-Titchmarsh condition* for $\sum_{n=1}^{\infty} f(nx)$ to be self-reciprocal is then that

$$\frac{\zeta(s)\,\phi(s)}{2^{\frac{1}{2}s}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}$$

should be an even function of $s-\frac{1}{2}$. Since $\Gamma(\frac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)$ is an even function of $s-\frac{1}{2}$, we have

$$\frac{(\frac{1}{2}\pi)^{\frac{1}{2}s}\phi(s)}{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}=\frac{(\frac{1}{2}\pi)^{\frac{1}{2}(1-s)}\phi(1-s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})},$$

i.e.
$$\phi(s)=(\tfrac{1}{2}\pi)^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})}\phi(1-s).$$

Hence

$$\begin{aligned}f(x)&=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}(\tfrac{1}{2}\pi)^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})}\phi(1-s)x^{-s}\,ds\\&=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})\,ds}{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})}\int_0^{\infty}(\tfrac{1}{2}\pi xy)^{-s}f(y)\,dy\\&=\int_0^{\infty}\varpi(xy)f(y)\,dy,\end{aligned}$$

where

$$\varpi(xy)=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})}(\tfrac{1}{2}\pi xy)^{-s}\,ds. \qquad (12)$$

We now have to evaluate the integral on the right. If it is taken for granted that the two sets of poles

$$s=-2n, \qquad s=-\nu-\tfrac{1}{2}-2n \qquad (n=0,1,2,3,\ldots)$$

lie on the left of the contour, we find, by calculating the residues at these poles, that

$$\begin{aligned}\varpi(xy)=&\frac{2^{\frac{1}{2}}\Gamma(\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{}_0F_3(\tfrac{1}{2},\tfrac{1}{2}\nu+\tfrac{3}{4},-\tfrac{1}{2}\nu+\tfrac{3}{4};\tfrac{1}{4}\pi^2x^2y^2)+\\&+\frac{(2\pi)^{\frac{1}{2}}\Gamma(-\frac{1}{2}\nu-\frac{1}{4})(\frac{1}{2}\pi xy)^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2}\nu+\frac{3}{4})\Gamma(\nu+1)}{}_0F_3(\nu+1,\tfrac{1}{2}\nu+\tfrac{3}{4},\tfrac{1}{2}\nu+\tfrac{5}{4};\tfrac{1}{4}\pi^2x^2y^2).\end{aligned}$$

By a formula due to Hanumanta Rao† the expression on the right may be replaced by an integral containing Bessel functions, so that

$$\varpi(xy)=2\int_0^{\infty}\sqrt{(xt)}J_{\nu}(xt)\cos\left(\frac{2\pi y}{t}\right)\frac{dt}{t}. \qquad (13)$$

* loc. cit., 198 and 208.

† *Messenger of Math.*, 47 (1918), 134–7. See my *Theory of Bessel Functions* (1922), 437.

Now that this result is discovered, it may be derived from (12) more quickly by observing that, when $\varpi(x)$ is defined by (12), we have

$$\int_0^\infty \sqrt{(xt)}J_\nu(xt)\varpi(t)\,dt$$

$$= \frac{1}{2\pi i}\int_0^\infty \int_{c-\infty i}^{c+\infty i} x^{\frac{1}{2}}J_\nu(xt)\,\frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})}(\tfrac{1}{2}\pi t)^{\frac{1}{2}-s}\,ds dt$$

$$= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \pi^{\frac{1}{2}}\left(\frac{x}{\pi}\right)^{s-1}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\,ds = \frac{2}{x}\cos\left(\frac{2\pi}{x}\right),$$

and therefore $\varpi(x)$ is the Hankel transform of $\frac{2}{x}\cos\frac{2\pi}{x}$. Consequently

$$\varpi(x) = 2\int_0^\infty \sqrt{(xt)}J_\nu(xt)\cos\left(\frac{2\pi}{t}\right)\frac{dt}{t},$$

whence (13) is obvious.

Therefore, if $\sum_{n=1}^{\infty} f(nx)$ is a self-reciprocal function, $f(x)$ satisfies the integral equation

$$f(x) = 2\int_0^\infty\int_0^\infty \sqrt{(xt)}J_\nu(xt)\cos\left(\frac{2\pi y}{t}\right)f(y)\,\frac{dt}{t}\,dy. \qquad (14)$$

It may be remarked that

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\zeta(s)\phi(s)x^{-s}\,ds$$

is not necessarily equal to $\sum_{n=1}^{\infty} f(nx)$, as has been taken for granted, because the pole of $\zeta(s)$ at $s=1$ may have the effect of adding an extra term to the series; and also modifications of the series like those introduced in (3) may be essential since it may be necessary to alter the value of c in Mellin's formula in order to make

$$\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} n^{-s}\phi(s)x^{-s}\,ds$$

convergent.

6. We next consider the integral equation which is satisfied by $f(x)$ when the series

$$f(x)-f(3x)+f(5x)-f(7x)+\ldots$$

is a self-reciprocal function in the Hankel transform of order ν. As before, write

$$\phi(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

and then the condition to be satisfied by $\phi(s)$ is that

$$\frac{L(s)\phi(s)}{2^{1s}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}$$

should be an even function of $s-\frac{1}{2}$, where, as usual,

$$L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \dots .$$

From the functional equation for $L(s)$ it follows that

$$\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})(4/\pi)^{\frac{1}{2}s} L(s)$$

is an even function of $s-\frac{1}{2}$ and therefore

$$\phi(s) = \left(\frac{8}{\pi}\right)^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2})\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(1-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})}\phi(1-s).$$

Hence, as in § 5,

$$f(x) = \int_0^\infty \varpi_1(xy) f(y)\, dy,$$

where

$$\varpi_1(xy) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \left(\frac{8}{\pi}\right)^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2})\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(1-\frac{1}{2}s)\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})} (xy)^{-s}\, ds.$$

To calculate the value of $\varpi_1(x)$ observe that

$$\int_0^\infty \sqrt{(xt)} J_\nu(xt)\varpi_1(t)\, dt = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \left(\frac{4}{\pi}\right)^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(1-\frac{1}{2}s)} x^{s-1}\, ds$$

$$= \frac{1}{x}\sin\left(\frac{\pi}{2x}\right).$$

Therefore $\varpi_1(x)$ is the Hankel transform of $\frac{1}{x}\sin\left(\frac{\pi}{2x}\right)$, and so

$$\varpi_1(xy) = \int_0^\infty \sqrt{(xt)} J_\nu(xt) \sin\left(\frac{\pi y}{2t}\right)\frac{dt}{t}. \tag{15}$$

Therefore, if $f(x) - f(3x) + f(5x) - \dots$ is a self-reciprocal function, $f(x)$ formally satisfies the integral equation

$$f(x) = \int_0^\infty \int_0^\infty \sqrt{(xt)} J_\nu(xt) \sin\left(\frac{\pi y}{2t}\right) f(y)\, \frac{dt}{t}\, dy. \tag{16}$$

7. An interesting example* is obtained by supposing that

$$\sum_{n=1}^{\infty} r(n) f(nx)$$

is self-reciprocal, where $r(n)$ denotes the number of distinct representations of n as the sum of two squares.

Since
$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = \zeta(s) L(s),$$

we find that $f(x)$ now satisfies the integral equation

$$f(x) = \int_0^{\infty} \varpi_2(xy) f(y)\, dy,$$

where

$$\varpi_2(x) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \left(\frac{2}{\pi}\right)^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})} x^{-s}\, ds.$$

It follows that

$$\int_0^{\infty} \sqrt{(xt)} J_\nu(xt) \varpi_2(t)\, dt = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \left(\frac{2}{\pi^2}\right)^{s-\frac{1}{2}} \frac{\Gamma(s)}{\Gamma(1-s)} x^{s-1}\, ds$$

$$= \frac{\pi}{x\sqrt{2}} J_0\left(\sqrt{\frac{2\pi^2}{x}}\right),$$

and hence
$$\varpi_2(xy) = \frac{\pi}{\sqrt{2}} \int_0^{\infty} \sqrt{(xt)} J_\nu(xt) J_0\left(\sqrt{\frac{2\pi^2 y}{t}}\right) \frac{dt}{t}.$$

8. We now return to the kernels introduced in § 5 and § 6. By taking Hanumanta Rao's formula

$$\int_0^{\infty} t^{\rho-1} J_\nu(at) J_\mu(b/t)\, dt$$

$$= \frac{a^{\mu-\rho} b^\mu \Gamma(\frac{1}{2}\nu+\frac{1}{2}\rho-\frac{1}{2}\mu)}{2^{2\mu-\rho+1}\Gamma(\mu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\rho+1)} \times$$

$$\times {}_0F_3\left(\mu+1, \frac{\mu-\nu-\rho}{2}+1, \frac{\mu+\nu-\rho}{2}+1; \frac{a^2b^2}{16}\right) +$$

$$+ \frac{a^\nu b^{\nu+\rho} \Gamma(\frac{1}{2}\mu-\frac{1}{2}\rho-\frac{1}{2}\nu)}{2^{2\nu+\rho+1}\Gamma(\nu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\rho+1)} \times$$

$$\times {}_0F_3\left(\nu+1, \frac{\nu-\mu+\rho}{2}+1, \frac{\mu+\nu+\rho}{2}+1; \frac{a^2b^2}{16}\right),$$

* Suggested by example 9 (p. 212) of the Hardy-Titchmarsh paper.

and using Barnes's asymptotic expansions* for the generalized hypergeometric functions on the right, it may be verified by some rather tedious analysis that, when a and b are positive and their product is large,

$$\int_0^\infty t^{\rho-1}J_\nu(at)J_\mu(b/t)\,dt = O(a^{-\frac{3}{4}-\frac{1}{2}\rho}b^{-\frac{3}{4}+\frac{1}{2}\rho}),$$

provided that μ, ν, ρ are such that

$$-R(\nu+\tfrac{3}{2}) < R(\rho) < R(\mu+\tfrac{3}{2}).$$

and so, when x is large and positive,

$$\varpi(x) = O(x^{-\frac{1}{4}}), \qquad \varpi_1(x) = O(x^{-\frac{1}{4}}).$$

Of course, each of the terms on the right in Hanumanta Rao's formula is exponentially large like

$$a^{-\frac{3}{4}-\frac{1}{2}\rho}b^{-\frac{3}{4}+\frac{1}{2}\rho}e^{2\sqrt{(ab)}}$$

when ab is large and positive, but the asymptotic series which have this positive exponential as a factor happen to cancel in the combination of the two functions ${}_0F_3$ which is under discussion.

Again it is easy to see that, when x is small and positive,

$$\varpi(x) = O(1)+O(x^{\nu+\frac{1}{2}}), \qquad \varpi_1(x) = O(x)+O(x^{\nu+\frac{1}{2}}).$$

It follows that, when $R(\nu+\frac{1}{2}) \geqslant 0$, there is no necessity to interpret the integral equations of § 5 and § 6,

$$f(x) = \int_0^\infty \varpi(xy)f(y)\,dy, \qquad f(x) = \int_0^\infty \varpi_1(xy)f(y)\,dy$$

in any Pickwickian sense, as would have been the case if the two series containing positive exponentials had not cancelled.

9. In conclusion I remark that Hanumanta Rao's integral with $\rho = 0$, but μ *not* equal to $\pm\frac{1}{2}$, as has hitherto been the case, can be regarded as a kernel which gives rise to a transform, at any rate formally. This transform has not, I think, been noticed previously and the kernel is a generalized hypergeometric function of a type quite different from the kernels discussed by Fox.†

Let $R(\mu+\frac{1}{2}) \geqslant 0$ and $R(\nu+\frac{1}{2}) \geqslant 0$ and define $\varpi_{\mu,\nu}(xy)$ by the integral

$$\varpi_{\mu,\nu}(xy) = A(xy)^{\frac{1}{2}}\int_0^\infty J_\nu(xt)J_\mu\left(\frac{Ay}{t}\right)\frac{dt}{t}$$

where A is any positive constant.

* *Proc. London Math. Soc.* (2) 5 (1907), 59–116.

† Ibid. (2) 29 (1929), 401–52.

It is obvious that

$$\varpi_{\mu,\nu}(xy) = A(xy)^{\frac{1}{2}} \int_0^\infty J_\nu(t) J_\mu\left(\frac{Axy}{t}\right) \frac{dt}{t}.$$

Let $f(x)$ be an arbitrary function, and let $g(x)$ be its transform with the kernel $\varpi_{\mu,\nu}(xy)$, so that

$$g(x) = \int_0^\infty \varpi_{\mu,\nu}(xy) f(y)\, dy.$$

Then, assuming that the various changes in the order of the integrations are permissible, we have

$$\begin{aligned}
&\int_0^\infty \varpi_{\mu,\nu}(xy) g(y)\, dy \\
&\quad = \int_0^\infty \int_0^\infty \varpi_{\mu,\nu}(xy) \varpi_{\mu,\nu}(yz) f(z)\, dz dy \\
&\quad = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty A^2 (xy^2 z)^{\frac{1}{2}} J_\nu(t) J_\nu(T) J_\mu\left(\frac{Axy}{t}\right) J_\mu\left(\frac{Ayz}{T}\right) f(z) \frac{dt}{t} \frac{dT}{T}\, dz dy.
\end{aligned}$$

Now, if we replace y by yT/A, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left(\frac{Axy}{t}\right)^{\frac{1}{2}} J_\mu\left(\frac{Axy}{t}\right) \left(\frac{Ayz}{T}\right)^{\frac{1}{2}} J_\mu\left(\frac{Ayz}{T}\right) f(z)\, dz dy \\
&\quad = \frac{T}{A} \int_0^\infty \int_0^\infty \left(\frac{xyT}{t}\right)^{\frac{1}{2}} J_\mu\left(\frac{xyT}{t}\right) (yz)^{\frac{1}{2}} J_\mu(yz) f(z)\, dz dy = \frac{T}{A} f\left(\frac{xT}{t}\right),
\end{aligned}$$

by Hankel's formula. Hence

$$\begin{aligned}
\int_0^\infty \varpi_{\mu,\nu}(xy) g(y)\, dy &= \int_0^\infty \int_0^\infty (tT)^{\frac{1}{2}} J_\nu(t) J_\nu(T) f\left(\frac{xT}{t}\right) \frac{dt dT}{t} \\
&= \int_0^\infty \int_0^\infty (uxT)^{\frac{1}{2}} J_\nu(ux) J_\nu(T) f\left(\frac{T}{u}\right) \frac{dT du}{u} \\
&= \int_0^\infty \int_0^\infty (ux)^{\frac{1}{2}} (uv)^{\frac{1}{2}} J_\nu(ux) J_\nu(uv) f(v)\, dv du \\
&= f(x),
\end{aligned}$$

again by Hankel's formula.

Consequently $f(x)$ and $g(x)$ are (formally) reciprocal functions in the transform with kernel $\varpi_{\mu,\nu}(xy)$.

The formulation of conditions with which the formal analysis is justifiable is a matter for further investigation.

ON A TAUBERIAN THEOREM CONCERNING DIRICHLET'S SERIES WITH POSITIVE COEFFICIENTS

By K. ANANDA-RAU (*Madras*)

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LET α, α_1, α_2,..., α_m be real constants not all zero, and let the first of them which is different from zero be greater than 0. Let us write for shortness $l_1 u = \log u$, $l_2 u = \log(l_1 u)$,..., $l_r u = \log(l_{r-1} u)$,...; and let $L(u) = (l_1 u)^{\alpha_1}(l_2 u)^{\alpha_2}...(l_m u)^{\alpha_m}$. From the conditions imposed on the α's it is clear that $u^{\alpha} L(u) \to \infty$ as $u \to \infty$. The following is a known theorem:

THEOREM A. *Suppose that* $a_n \geqslant 0$ *for* $n = 1, 2, 3,...,$ *and that*

$$f(s) = \sum_1^{\infty} a_n e^{-\lambda_n s}$$

is a Dirichlet's series convergent for $s > 0$*; suppose further that when* $s \to 0$

$$f(s) \sim \frac{A}{s^{\alpha}} L\left(\frac{1}{s}\right) \tag{1}$$

where $A > 0$ *is a constant. Then as* $n \to \infty$,

$$A_n = a_1 + a_2 + ... + a_n \sim \frac{A \lambda_n^{\alpha} L(\lambda_n)}{\Gamma(\alpha+1)}. \tag{2}$$

The above theorem, with the restriction $\lambda_{n+1} \sim \lambda_n$, is due to Hardy and Littlewood.* It has been recently shown that it is not necessary to include any explicit restriction on the λ's in the hypothesis of the theorem.† The object of the present note is to deduce Theorem A in its general form for any Dirichlet's series from the particular case of the theorem for power series, that is, the case when $\lambda_n = n$.

* *Messenger of Math.*, 43 (1914), 134–47 (141–6). In an earlier paper Hardy and Littlewood gave a proof of the particular case of Theorem A for power series: see *Proc. London Math. Soc.* (2) 13 (1913), 174–91 (180–5).

† O. Szász, 'Verallgemeinerung und neuer Beweis einiger Satze Tauberscher Art', *Sitzungsb. der Bayer. Akad.* (Math. Naturwiss. Abt.), 1929, pp. 325–40.

J. Karamata, 'Neuer Beweis und Verallgemeinerung einiger Tauberian-Sätze', *Math. Zeitschrift*, 33 (1931), 294–9.

K. Ananda-Rau, 'On Dirichlet's series with positive coefficients', *Rend. di Palermo*, 54 (1930), 455–61.

I may take this opportunity of pointing out that the proof given in my paper in the *Palermo Rendiconti* referred to above can be easily adapted to

To do this let $p = p(n)$, $q = q(n)$ be defined by

$$p-1 \leqslant \lambda_n < p, \tag{3}$$

$$\lambda_{n+q} < p \leqslant \lambda_{n+q+1}, \tag{4}$$

so that $q \geqslant 0$. Further, let

$$c_n = \sum_{n-1 \leqslant \lambda_r < n} a_r \qquad (n = 1, 2, 3, \ldots),$$

$$F(s) = \sum_1^\infty c_n e^{-ns},$$

$$C_r = c_1 + c_2 + \ldots + c_r.$$

We have* $$c_n e^{-ns} < \sum_{n-1 \leqslant \lambda_r < n} a_r e^{-\lambda_r s} \leqslant c_n e^{-(n-1)s},$$

$$\sum_{n=1}^\infty c_n e^{-ns} < \sum_{r=1}^\infty a_r e^{-\lambda_r s} \leqslant e^s \sum_{n=1}^\infty c_n e^{-ns},$$

$$F(s) < f(s) \leqslant e^s F(s),$$

from which it follows that when $s \to 0$

$$F(s) \sim f(s),$$

and so by (1) $$F(s) \sim \frac{A}{s^\alpha} L\left(\frac{1}{s}\right).$$

Now $F(s)$ is a power series in e^{-s} with positive coefficients, and we apply the particular case of Theorem A when $\lambda_n = n$. We then get, corresponding to (2),

$$C_p \sim \frac{A p^\alpha L(p)}{\Gamma(\alpha+1)} \tag{5}$$

when $p \to \infty$. Since $p \to \infty$ as $n \to \infty$, (5) is true when $n \to \infty$.

prove the following theorem, which may be considered as completing the earlier one.

THEOREM. *Suppose that $a_n \geqslant 0$ for $n = 1, 2, 3, \ldots$, and that*

$$f(s) = \sum_1^\infty a_n e^{-\lambda_n s}$$

is convergent for $s > 0$; suppose further that, as $s \to 0$,

$$f(s) \sim AL\left(\frac{1}{s}\right),$$

where $A > 0$ is a constant and $L(u) = (l_1 u)^{\alpha_1} (l_2 u)^{\alpha_2} \ldots (l_m u)^{\alpha_m}$ tends to infinity when $u \to \infty$. Let the first of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ which is different from zero (and therefore greater than zero) be α_k. Then, as $n \to \infty$,

$$l_k(\lambda_{n+1}) \sim l_k(\lambda_n).$$

This theorem can also be deduced from Satz 1 of Karamata's paper referred to above.

* Cf. K. Knopp, 'Grenzwerte von Dirichletschen Reihen bei der Annäherung an die Konvergenzgrenze': *J. für Math.* 138 (1910), 109–32 (115–16).

Now by (4)

$$C_p = \sum_{\lambda_r < p} a_r = A_{n+q} = A_n + (a_{n+1} + \ldots + a_{n+q}) \tag{6}$$

with the understanding that when n is such that $q = q(n) = 0$, the expression $(a_{n+1} + \ldots + a_{n+q})$ in (6) and in the following is to be replaced by zero. Since $p - 1 \to \infty$ with n, we have, similarly to (5),

$$C_{p-1} \sim \frac{A(p-1)^\alpha L(p-1)}{\Gamma(\alpha+1)} \sim \frac{Ap^\alpha L(p)}{\Gamma(\alpha+1)},$$

and so

$$c_p = C_p - C_{p-1} = o\{p^\alpha L(p)\}. \tag{7}$$

On the other hand, in view of (3) and (4),

$$c_p = \sum_{p-1 \leqslant \lambda_r < p} a_r \geqslant a_{n+1} + \ldots + a_{n+q},$$

and so, by applying (7),

$$a_{n+1} + \ldots + a_{n+q} = o\{p^\alpha L(p)\}. \tag{8}$$

Therefore, combining (5), (6), (8), we get

$$\begin{aligned} A_n &= C_p - (a_{n+1} + \ldots + a_{n+q}) \\ &\sim \frac{Ap^\alpha L(p)}{\Gamma(\alpha+1)} \sim \frac{A\lambda_n^\alpha L(\lambda_n)}{\Gamma(\alpha+1)}, \end{aligned}$$

on making use of the relation $p \sim \lambda_n$ obtained from (3).

ON VAN DER CORPUT'S METHOD AND THE ZETA-FUNCTION OF RIEMANN (II)

By E. C. TITCHMARSH (*Oxford*)

[Received 2 October 1931]

1. In my previous paper* (which will be referred to as I) I gave a simplified form of the proof of the theorem of van der Corput and Koksma that, on the lines $\sigma = 1-(q+1)/(4Q-2)$, where $Q = 2^{q-1}$, $q = 2, 3, \ldots$,

$$\zeta(\sigma+it) = O\left(t^{\frac{1}{4Q-2}}\log t\right). \tag{1}$$

In particular, taking $q = 2$, we have

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{6}}\log t). \tag{2}$$

It is, however, known that

$$\zeta(\tfrac{1}{2}+it) = O(t^{\theta}), \qquad \theta < \tfrac{1}{6}, \tag{3}$$

—in fact van der Corput and Walfisz† proved (3) with $\theta = \frac{163}{988}$. In the present paper we give a simpler proof of (3); and, corresponding to van der Corput's later result for the divisor problem, we obtain a slightly smaller value of θ, viz. $\theta = \frac{27}{164}$.

2. The problem is to obtain suitable inequalities for sums of the form

$$\sum e^{2\pi i f(n)},$$

where $f(n)$ is a real function. There are two processes by which such a sum may be made to depend on another sum, of the same general form but with a different function instead of $f(n)$. One process, due in its original form to Weyl, is Lemma 3 of I. The other is an 'approximate functional equation', and is Theorem 1 of van der Corput (**2**). Let us denote these two processes by A and B respectively. Lemmas 1 and 2 of I are an incomplete form of B, say B'. The proof of (2) given in I can be expressed by

$$AB',$$

i.e. we first transform the given sum by Weyl's process, and then use Lemmas 1 and 2.

The proof of (3) due to van der Corput and Walfisz would be denoted in the same way by

$$ABAAABABAB.$$

* See above, pp. 161–73.

† See the references at the end of the previous paper.

The proof of (3) given below would be denoted by

$$ABAAAAB',$$

the $AAAAB'$ being given by Theorem 3 of I with $k=4$.

All that we require to know of process A is given in paper I. For the sake of completeness we shall now give the proof of B. We continue the system of numbering lemmas and theorems from I.

3. LEMMA 4. *Let $f(x)$ be a real function with a continuous and steadily decreasing differential coefficient $f'(x)$ in the interval (a,b), and let $f'(b)=\alpha$, $f'(a)=\beta$. Then*

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = \sum_{\alpha-1<\nu<\beta+1} \int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx + O\{\log(\beta-\alpha+2)\}. \qquad (1)$$

The proof is similar to that of Lemma 1.

We may suppose without loss of generality that $0\leqslant\alpha<1$; for otherwise let r be the integer such that $\alpha-1<r\leqslant\alpha$, and let

$$g(x)=f(x)-rx, \qquad \alpha'=g'(b)=\alpha-r, \qquad \beta'=g'(a)=\beta-r.$$

Then (1) is

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i g(n)} = \sum_{\alpha'-1<\nu<\beta'+1} \int_a^b e^{2\pi i\{g(x)-\nu x\}}\,dx + O\{\log(\beta'-\alpha'+2)\},$$

i.e. the same formula for $g(x)$; and $0\leqslant\alpha'<1$.

With the notation of Lemma 1, we have

$$2\pi i\int_a^b \chi(x)e^{2\pi i f(x)}f'(x)\,dx = -\sum_{\nu=1}^{\infty}\frac{2i}{\nu}\int_a^b \sin 2\nu\pi x\, e^{2\pi i f(x)}f'(x)\,dx \qquad (2)$$

$$= -\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_a^b e^{2\pi i\{f(x)+\nu x\}}f'(x)\,dx + \sum_{\nu<\beta+1}\frac{1}{\nu}\int_a^b e^{2\pi i\{f(x)-\nu x\}}\{f'(x)-\nu\}\,dx +$$

$$+\sum_{\nu<\beta+1}\int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx + \sum_{\nu\geqslant\beta+1}\frac{1}{\nu}\int_a^b e^{2\pi i\{f(x)-\nu x\}}f'(x)\,dx$$

$$=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,$$

say. Since $f'(x)/\{f'(x)+\nu\}$ is monotonic, the second mean-value theorem gives

$$\Sigma_1 = -\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_a^b \frac{f'(x)}{f'(x)+\nu}\,d\,e^{2\pi i\{f(x)+\nu x\}}$$

$$= O\Big(\sum_{\nu=1}^{\infty}\frac{1}{\nu}\frac{\beta}{\beta+\nu}\Big) = O\Big\{\frac{\beta}{\beta+1}+\int_1^{\infty}\frac{\beta\,dx}{x(\beta+x)}\Big\} = O\{\log(\beta+1)\}.$$

Similarly

$$\Sigma_4 = \sum_{\nu \geqslant \beta+1}^{\infty} \frac{1}{\nu} \int_a^b \frac{f'(x)}{f'(x)-\nu}\, d\, e^{2\pi i\{f(x)-\nu x\}}$$

$$= O\Big(\sum_{\nu \geqslant \beta+1}^{\infty} \frac{1}{\nu} \frac{\beta}{\nu-\beta} \Big) = O\{\log(\beta+1)\}.$$

Also $$\Sigma_2 = \sum_{\nu<\beta+1} \frac{1}{\nu} \Big[\frac{e^{2\pi i\{f(x)-\nu x\}}}{2\pi i} \Big]_a^b = \sum_{\nu<\beta+1} O\Big(\frac{1}{\nu}\Big) = O\{\log(\beta+1)\}.$$

As in the proof of Lemma 1, the left-hand side of (2) is equal to

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} - \int_a^b e^{2\pi i f(x)}\, dx + O(1).$$

Hence

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\, dx + \sum_{0<\nu<\beta+1} \int_a^b e^{2\pi i\{f(x)-\nu x\}}\, dx + O\{\log(\beta+1)\} + O(1),$$

and, since $0 \leqslant \alpha < 1$, the result follows.

4. THEOREM 4. *Let $f(x)$ satisfy the conditions of Lemma 4, and let $f''(x)$ and $f'''(x)$ be continuous, and*

$$m_2 \leqslant |f''(x)| < A m_2, \qquad |f'''(x)| < A m_3,$$

for $a \leqslant x \leqslant b$. Let n_ν be such that $f'(n_\nu) = \nu$ $(\alpha \leqslant \nu \leqslant \beta)$. Then

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = e^{-\frac{1}{4}\pi i} \sum_{\alpha \leqslant \nu \leqslant \beta} \frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}}}{|f''(n_\nu)|^{\frac{1}{2}}} + O(m_2^{-\frac{1}{2}}) + $$

$$+ O[\log\{2+(b-a)m_2\}] + O\{(b-a)m_2^{\frac{1}{2}} m_3^{\frac{1}{3}}\}. \qquad (1)$$

Starting from the result of Lemma 4, we may replace the sum on the right of § 3 (1) by

$$\sum_{\alpha+1<\nu<\beta-1} \int_a^b e^{2\pi i\{f(x)-\nu x\}}\, dx$$

with error $O(m_2^{-\frac{1}{2}})$; for we omit four terms, each of which is $O(m_2^{-\frac{1}{2}})$, by Lemma 2. The new sum is empty unless $\beta-\alpha > 2$, so that we now suppose that this condition is satisfied. We write

$$\int_a^b e^{2\pi i\{f(x)-\nu x\}}\, dx = \int_a^{n_\nu-\delta} + \int_{n_\nu-\delta}^{n_\nu+\delta} + \int_{n_\nu+\delta}^b = J_1 + J_2 + J_3.$$

Then

$$J_3 = \int_{n_\nu+\delta}^{b} \frac{d\, e^{2\pi i\{f(x)-\nu x\}}}{2\pi i\{f'(x)-\nu\}} = O\left\{\frac{1}{\nu-f'(n_\nu+\delta)}\right\}$$

$$= O\left\{\frac{1}{\delta|f''(n_\nu+\theta\delta)|}\right\} = O\left(\frac{1}{\delta m_2}\right),$$

and a similar result holds for J_1. Also

$$J_2 = \int_{-\delta}^{\delta} e^{2\pi i\{f(n_\nu+t)-\nu n_\nu-\nu t\}}\, dt = \int_{\delta}^{-\delta} e^{2\pi i\{f(n_\nu)-\nu n_\nu+\frac{1}{2}t^2 f''(n_\nu)+\frac{1}{6}t^3 f'''(n_\nu+\theta t)\}}\, dt$$

$$= e^{2\pi i\{f(n_\nu)-\nu n_\nu\}} \int_{-\delta}^{\delta} e^{\pi i t^2 f''(n_\nu)}\{1+O(t^3 m_3)\}\, dt$$

$$= e^{2\pi i\{f(n_\nu)-\nu n_\nu\}} \int_{-\delta}^{\delta} e^{\pi i t^2 f''(n_\nu)}\, dt + O(\delta^4 m_3),$$

and, since $f''(n_\nu) < 0$,

$$\int_{-\delta}^{\delta} e^{\pi i t^2 f''(n_\nu)}\, dt = \frac{1}{\{\pi|f''(n_\nu)|\}^{\frac{1}{2}}} \int_{0}^{\pi\delta^2|f''(n_\nu)|} \frac{e^{-iu}}{\sqrt{u}}\, du$$

$$= \frac{e^{-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}} - \frac{1}{\{\pi|f''(n_\nu)|\}^{\frac{1}{2}}} \int_{\pi\delta^2|f''(n_\nu)|}^{\infty} \frac{e^{-iu}}{\sqrt{u}}\, du = \frac{e^{-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}} + O\left(\frac{1}{\delta m_2}\right).$$

Altogether we have

$$\int_a^b e^{2\pi i\{f(x)-\nu x\}}\, dx = \frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}} + O\left(\frac{1}{\delta m_2}\right) + O(\delta^4 m_3).$$

The two O-terms are of the same order if $\delta = (m_2 m_3)^{-\frac{1}{5}}$; and if this value of δ is such that

$$a \leqslant n_\nu - \delta, \qquad n_\nu + \delta \leqslant b, \tag{2}$$

we obtain

$$\int_a^b e^{2\pi i\{f(x)-\nu x\}}\, dx = \frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}} + O(m_2^{-\frac{1}{5}} m_3^{\frac{1}{3}}).$$

If (2) is satisfied for all values of ν such that $\alpha+1 < \nu < \beta-1$, we obtain, on summing with respect to ν,

$$\sum_{\alpha+1<\nu<\beta-1} \int_a^b e^{2\pi i\{f(x)-\nu x\}}\, dx$$

$$= e^{-\frac{1}{4}\pi i} \sum_{\alpha+1<\nu<\beta-1} \frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}}}{|f''(n_\nu)|^{\frac{1}{2}}} + O\{(\beta-\alpha) m_2^{-\frac{1}{5}} m_3^{\frac{1}{3}}\}. \tag{4}$$

The limits of the sum on the right may be replaced by (α, β), with error $O(m_2^{-1})$; and

$$\beta-\alpha = f'(a)-f'(b) = O\{(b-a)m_2\},$$

so that the last term in (4) is $O\{(b-a)m_2^{\frac{1}{2}}m_3^{\frac{1}{3}}\}$. The theorem then follows from Lemma 4.

Suppose next that there are values of ν for which $n_\nu+\delta > b$. The argument proceeds as before, except that we now obtain a term

$$\int_{-\delta}^{b-n_\nu} e^{\pi i t^2 f''(n_\nu)}\, dt$$

instead of the corresponding integral over $(-\delta, \delta)$. We therefore have to consider an additional error

$$\int_{b-n_\nu}^{\delta} e^{\pi i t^2 f''(n_\nu)}\, dt = O\{m_2^{-1}(b-n_\nu)^{-1}\}.$$

If $\psi(x)$ is the inverse function of $f'(x)$, so that $|\psi'(x)| = |f''(x)|^{-1} \geqslant 1/m_2$, then $\qquad b-n_\nu = \psi(\alpha)-\psi(\nu) = (\alpha-\nu)\psi'(\xi) \geqslant (\nu-\alpha)/m_2.$

Hence the sum of these terms is

$$O\Big(\sum_{\alpha+1<\nu<\beta-1} \frac{1}{\nu-\alpha}\Big) = O\{\log(\beta-\alpha)\} = O[\log\{2+(b-a)m_2\}],$$

and the theorem again follows. A similar argument disposes of the case $n_\nu-\delta < a$.

5. THEOREM. $\qquad \zeta(\tfrac{1}{2}+it) = O\Big(t^{\frac{27}{164}}\Big).$

Consider the sum

$$\Sigma_1 = \sum_{n=a}^{b} n^{-it} = \sum_{n=a}^{b} e^{-it\log n} \qquad (a < b \leqslant 2a).$$

By Lemma 3

$$\Sigma_1 = O\Big(\frac{a}{\rho^{\frac{1}{2}}}\Big) + O\Big\{\Big(\frac{a}{\rho}\sum_{s=1}^{\rho-1}\Big|\sum\nolimits_2\Big|\Big)^{\frac{1}{2}}\Big\}, \tag{1}$$

where $\qquad \rho \leqslant b-a,$ and $\Sigma_2 = \sum_{n=a}^{b-s} e^{-it\{\log(n+s)-\log n\}}.$

We now apply Theorem 4 to Σ_2. We have

$$f(x) = -\frac{t}{2\pi}\{\log(x+s)-\log x\}, \qquad f'(x) = \frac{st}{2\pi x(x+s)},$$

$$f''(x) = -\frac{st}{2\pi}\frac{2x+s}{x^2(x+s)^2}, \qquad f'''(x) = \frac{st}{\pi}\frac{3x^2+3xs+s^2}{x^3(x+s)^3}.$$

Since $s<\rho\leqslant b-a\leqslant a$, and $x\geqslant a$, we have

$$Asta^{-3}<|f''(x)|<Asta^{-3}, \qquad |f'''(x)|<Asta^{-4}.$$

Theorem 4 therefore gives

$$\sum\nolimits_2 = e^{-\frac{1}{4}\pi i}\sum_{\alpha\leqslant\nu\leqslant\beta}\frac{e^{2\pi i\phi(\nu)}}{|f''(n_\nu)|^{\frac{1}{2}}}+O\left\{\frac{a^{\frac{1}{2}}}{(st)^{\frac{1}{2}}}\right\}+O\left\{\log\left(2+\frac{st}{a^2}\right)\right\}+O\left\{\frac{(st)^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}, \quad (2)$$

where $\phi(\nu)=f(n_\nu)-\nu n_\nu$, and

$$\alpha=f'(b-s)=\frac{st}{2\pi(b-s)b}, \qquad \beta=f'(a)=\frac{st}{2\pi a(a+s)}.$$

We next consider the sum

$$\sum_{\alpha\leqslant\nu\leqslant\gamma}e^{2\pi i\phi(\nu)} \qquad (\alpha<\gamma\leqslant\beta).$$

The numbers n_ν are given by

$$\frac{st}{2\pi n_\nu(n_\nu+s)}=\nu, \qquad \text{i.e. } n_\nu=\frac{1}{2}\left(s^2+\frac{2st}{\pi\nu}\right)^{\frac{1}{2}}-\tfrac{1}{2}s.$$

Hence

$$\phi'(\nu)=\{f'(n_\nu)-\nu\}\frac{dn_\nu}{d\nu}-n_\nu=-n_\nu=\tfrac{1}{2}s-\frac{1}{2}\left(s^2+\frac{2st}{\pi\nu}\right)^{\frac{1}{2}},$$

$$\phi''(\nu)=\frac{st}{2\pi\nu^2(s^2+2st/\pi\nu)^{\frac{1}{2}}}=\frac{(st)^{\frac{1}{2}}}{2(2\pi)^{\frac{1}{2}}}\left(\frac{1}{\nu^{\frac{3}{2}}}-\frac{\pi s}{4t}\frac{1}{\nu^{\frac{1}{2}}}+\ldots\right);$$

and if $a<A\sqrt{t}$, so that $s/t<A/\sqrt{t}$, it is clear that

$$\frac{K_1(st)^{\frac{1}{2}}}{\nu^{k+\frac{1}{2}}}<|\phi^{(k+1)}(\nu)|<\frac{K_2(st)^{\frac{1}{2}}}{\nu^{k+\frac{1}{2}}} \qquad (t>t_k),$$

K_1, K_2, and t_k depending on k only.

We now use Theorem 3. We have

$$R_k=O\{(st)^{\frac{1}{2}}\alpha^{-(k+\frac{1}{2})}\}=O\{a^{2k+1}(st)^{-k}\},$$

the constant implied in the O depending on k only; and a similar result holds for $1/r_k$. Hence

$$\sum_{\alpha\leqslant\nu<\gamma}e^{2\pi i\phi(\nu)}=O\left[\frac{st}{a^2}\left\{\frac{a^{2k+1}}{(st)^k}\right\}^{\frac{1}{4K-2}}\right]+O\left[\left(\frac{st}{a^2}\right)^{1-\frac{1}{K}}\left\{\frac{a^{2k+1}}{(st)^k}\right\}^{-\frac{1}{4K-2}}\right]$$

$$=O\left\{(st)^{1-\frac{k}{4K-2}}a^{\frac{2k+1}{4K-2}-2}\right\}+O\left\{(st)^{1-\frac{1}{K}+\frac{k}{4K-2}}a^{\frac{2}{K}-2-\frac{2k+1}{4K-2}}\right\}.$$

Also $|f''(n_\nu)|^{-\frac{1}{2}}$ is monotonic and of the form $O\{(st)^{-\frac{1}{4}}a^{\frac{1}{2}}\}$. Hence, by partial summation,

$$\sum_{\alpha\leqslant\nu\leqslant\beta}\frac{e^{2\pi i\phi(\nu)}}{|f''(n_\nu)|^{\frac{1}{2}}}=O\left\{(st)^{\frac{1}{2}-\frac{k}{4K-2}}a^{\frac{2k+1}{4K-2}-\frac{1}{2}}\right\}+$$

$$+O\left\{(st)^{\frac{1}{2}-\frac{1}{K}+\frac{k}{4K-2}}a^{\frac{2}{K}-\frac{1}{2}-\frac{2k+1}{4K-2}}\right\}. \quad (3)$$

We now insert this in (2). Observing that the logarithmic term in (2) is always of smaller order than one of the other terms, we obtain

$$\Sigma_2 = O\left\{(st)^{\frac{1}{2}-\frac{k}{4K-2}} a^{\frac{2k+1}{4K-2}-\frac{1}{2}}\right\} + O\left\{(st)^{\frac{1}{2}-\frac{1}{K}+\frac{k}{4K-2}} a^{\frac{2}{K}-\frac{1}{2}-\frac{2k+1}{4K-2}}\right\} + \\ + O\{(st)^{-\frac{1}{2}}a^{\frac{1}{2}}\} + O\{(st)^{\frac{1}{2}}a^{-\frac{1}{2}}\}. \quad (4)$$

Finally we insert this in (1). Summing with respect to ρ and dividing by ρ has the effect of replacing s by ρ in each term. We then multiply by a; and we can take the square root of each term separately, since

$$(X+Y+\ldots)^{\frac{1}{2}} \leqslant X^{\frac{1}{2}}+Y^{\frac{1}{2}}+\ldots.$$

The result is

$$\Sigma_1 = O(\rho^{-\frac{1}{2}}a) + O\left\{(\rho t)^{\frac{1}{4}-\frac{k}{8K-4}} a^{\frac{2k+1}{8K-4}+\frac{1}{4}}\right\} + \\ + O\left\{(\rho t)^{\frac{1}{4}-\frac{1}{2K}+\frac{k}{8K-4}} a^{\frac{1}{K}+\frac{1}{4}-\frac{2k+1}{8K-4}}\right\} + O\{(\rho t)^{-\frac{1}{2}}a^{\frac{1}{2}}\} + O\{(\rho t)^{\frac{1}{2}}a^{\frac{1}{2}}\}. \quad (5)$$

The first two terms on the right of (5) are of the same order if

$$\rho = \left[a^{\frac{6K-2k-4}{6K-k-3}} t^{-\frac{2K-k-1}{6K-k-3}}\right] \qquad (1 \leqslant \rho \leqslant b-a),$$

and they are then of the form

$$O\left(a^{\frac{3K-1}{6K-k-3}} t^{\frac{1}{2}\frac{2K-k-1}{6K-k-3}}\right).$$

We have to take a as large as $A\sqrt{t}$, and they are then of the form

$$O\left(t^{\frac{5K-k-2}{2(6K-k-3)}}\right).$$

For $k = 1, 2, 3, 4, 5,\ldots$ the index has the values

$$\frac{1}{2}, \quad \frac{3}{7}, \quad \frac{5}{12}, \quad \frac{17}{41}, \quad \frac{73}{176}, \ldots,$$

and of these $\frac{17}{41}$ is the least. We therefore take $k = 4$. Then

$$\rho = \left[a^{\frac{36}{41}} t^{-\frac{11}{41}}\right] \qquad \left(a > t^{\frac{11}{36}}\right),$$

and we obtain

$$\Sigma_1 = O\left(a^{\frac{23}{41}} t^{\frac{11}{82}}\right) + O\left(a^{\frac{147}{328}} t^{\frac{61}{328}}\right) + O\left(a^{\frac{169}{164}} t^{-\frac{15}{82}}\right) + O\left(a^{\frac{39}{82}} t^{\frac{6}{41}}\right). \quad (6)$$

This also holds if $\rho \geqslant b-a$, since then

$$\Sigma_1 = O(b-a) = O(\rho) = O\left(a^{\frac{36}{41}} t^{-\frac{11}{41}}\right),$$

which is of smaller order than the third term on the right of (6).

It follows from (6) by partial summation that, for $b \leqslant 2a$, $a > t^{11/36}$,

$$\sum_{n=a}^{b} \frac{1}{n^{\frac{1}{2}+it}} = O\left(a^{\frac{5}{82}} t^{\frac{11}{82}}\right) + O\left(a^{-\frac{17}{328}} t^{\frac{61}{328}}\right) + O\left(a^{\frac{87}{164}} t^{-\frac{15}{82}}\right) + O\left(a^{-\frac{1}{41}} t^{\frac{6}{41}}\right) \quad (7)$$

Applying this with $a=N$, $b=2N-1$; $a=2N$, $b=4N-1$,..., the last value of b being N', and adding, we obtain, for $N>t^{11/36}$,

$$\sum_{n=N}^{N'}\frac{1}{n^{\frac{1}{2}+it}}=O\left(N'^{\frac{5}{82}}t^{\frac{11}{82}}\right)+O\left(N^{-\frac{17}{328}}t^{\frac{61}{328}}\right)+O\left(N'^{\frac{87}{164}}t^{-\frac{15}{82}}\right)+O\left(N^{-\frac{1}{41}}t^{\frac{6}{41}}\right). \tag{8}$$

If $N'<A\sqrt{t}$, the first term on the right is $O\left(t^{\frac{27}{164}}\right)$. The second term is also of this form if $N>t^{7/17}$. The remaining terms are then negligible, and we obtain

$$\sum_{t^{\frac{7}{17}}<n<At^{\frac{1}{2}}}\frac{1}{n^{\frac{1}{2}+it}}=O\left(t^{\frac{27}{164}}\right). \tag{9}$$

We require a subsidiary argument for the corresponding sum with $n\leqslant t^{7/17}$, and Theorem 3 is sufficient for this purpose. In fact (1) of I, § 4.1, with $k=3$, gives

$$\sum_{n=N}^{N'}\frac{1}{n^{\frac{1}{2}+it}}=O\left(N^{\frac{3}{14}}t^{\frac{1}{14}}\right)\qquad\left(N\leqslant t^{\frac{4}{9}}\right).$$

Hence

$$\sum_{n=N}^{N'}\frac{1}{n^{\frac{1}{2}+it}}=O\left(t^{\frac{19}{119}}\right)\qquad\left(N<At^{\frac{7}{17}}\right),$$

and, summing in the usual way,

$$\sum_{n\leqslant t^{\frac{7}{17}}}\frac{1}{n^{\frac{1}{2}+it}}=O\left(t^{\frac{19}{119}}\log t\right)=O\left(t^{\frac{27}{164}}\right). \tag{10}$$

Hence

$$\sum_{n<At^{\frac{1}{2}}}\frac{1}{n^{\frac{1}{2}+it}}=O\left(t^{\frac{27}{164}}\right).$$

The result now follows from the approximate functional equation* of Hardy and Littlewood

$$\zeta(\tfrac{1}{2}+it)=\sum_{n<\sqrt{(\frac{t}{2\pi})}}\frac{1}{n^{\frac{1}{2}-it}}+\chi\sum_{n<\sqrt{(\frac{t}{2\pi})}}\frac{1}{n^{\frac{1}{2}+it}}+O(t^{-\frac{1}{4}}),$$

where $\chi=O(1)$.

* See E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 32–4, for a proof, and references to other proofs. The proof there given should be corrected by taking the integral referred to at the top of p. 34 along $(x, x+e^{\frac{1}{4}i\pi}\infty)$ instead of $(x, x+i\infty)$.

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